Many weak P-sets

Jan van Mill[1]

University of Amsterdam

Thirteenth Symposium on General Topology
and its Relations to Modern Analysis and Algebra
July 25, 2022

[1] Joint work with Alan Dow

TOPOSYM 2022

1961 1966 1971 **1976** 1981 1986 1991 1996 2001 2006 2011 2016 Inventory

Fourth Symposium on General Topology

and its Relations to Modern Analysis and Algebra

was held on August 23 27, 1976 in Prague, Czech Republic. It was organized by the Mathematical Institute of the **Czechoslovak Academy of Sciences** with support of the **International Mathematical Union** and in cooperation with the **Slovak Academy of Sciences**, the **Faculty of Mathematics and Physics** of the **Charles University** and the **Association of Czechoslovak Mathematicians and Physicists**.

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The Symposium was attended by 217 mathematicians from 24 countries, including 53 from Czechoslovakia. The program consisted of 30 invited talks (11 plenary, 18 semiplenary, 1 in a session for contributed papers), and 135 fifteen minute talks in three or four parallel sessions.

Prague 1976

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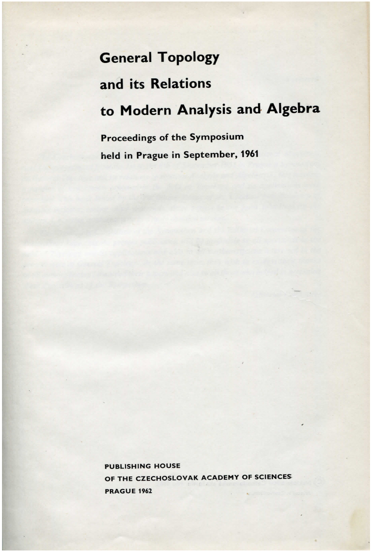
General Topology
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Proceedings of the Symposium
held in Prague in September, 1961

PUBLISHING HOUSE
OF THE CZECHOSLOVAK ACADEMY OF SCIENCES
PRAGUE 1962

1961

61 years ago!


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Isbell, Klee, Wallace (USA)
Lejeune, Hakim (France)
Nagata, Shirota (Japan)
Chogoshvili (Georgia)

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The Symposium was held in an atmosphere of friendship and contributed to the establishment and strengthening of personal contacts between the scientists from different countries.

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REPORT OF THE ORGANIZING COMMITTEE 9

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- *Prague was (and stayed) the perfect bridge between the East and the West, it brought people together in a divided world 61 years ago!*
- *Let us express hope that the war in Ukraine will not result in such a division again!*

Bohuslav Balcar

Petr Simon

Vera Trnková

Lev Bukovský

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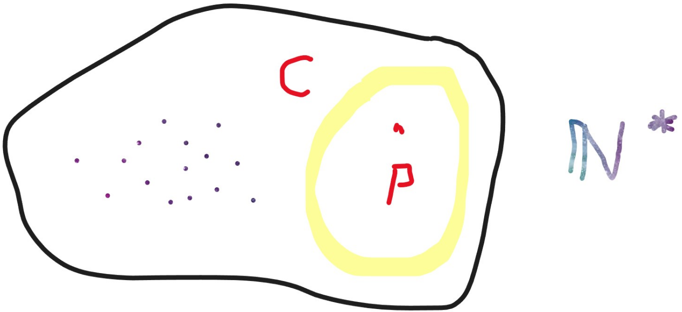
- $\beta\mathbb{N}$ surfaces at many places in mathematics: topology, set theory, logic, analysis, algebra, etc.
- In the 'old' days there was a lot of interest in the individual *points* of $\beta\mathbb{N}$.
- Walter Rudin proved that $\mathbb{N}^*$ is not *homogeneous* under CH. That is, there are two points in $\mathbb{N}^*$ that have different topological behavior in $\mathbb{N}^*$. Frolík proved this in ZFC. Shelah proved that Rudin's method does not work in ZFC alone.

- A definitive result was proved by Kunen in 1978: $\mathbb{N}^*$ contains a so-called *weak P-point*. That is a point $p \in \mathbb{N}^*$ such that $p \notin \overline{A}$, where A is any countable subset of $\mathbb{N}^* \setminus \{p\}$.



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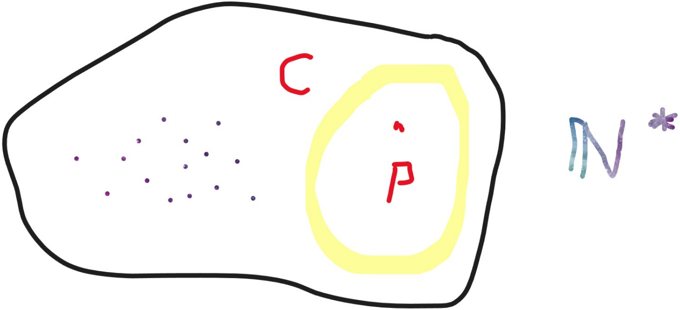

- If $A \subseteq \mathbb{N}^*$ is any countably infinite set, then there exists $q \in \overline{A} \setminus A$, hence q is not a weak P-point.

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- Are there such subspaces, besides $\mathbb{N}^*$ itself?

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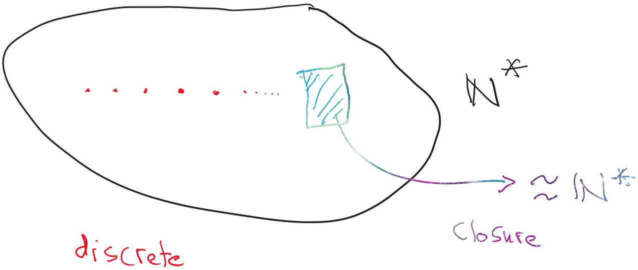

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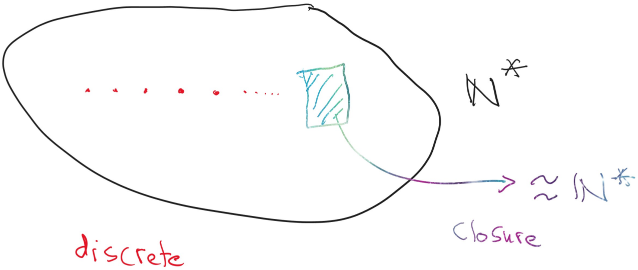

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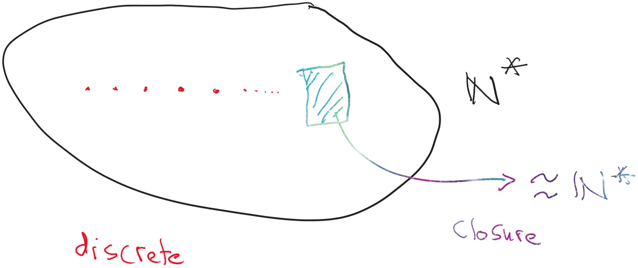

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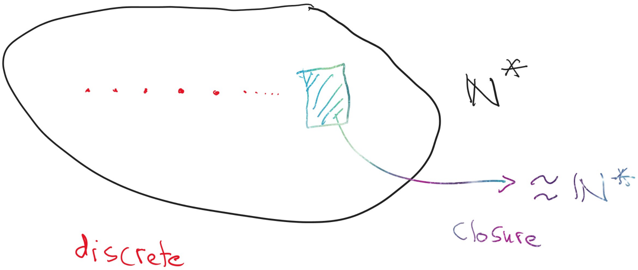

- van Douwen called such copies of $\mathbb{N}^*$ in $\mathbb{N}^*$ *trivial*.
- Around 1980 (our best guess) he asked: *is there a nowhere dense copy of $\mathbb{N}^*$ in $\mathbb{N}^*$ that is not trivial?*
- Reformulating: is there a nowhere dense copy of $\mathbb{N}^*$ in $\mathbb{N}^*$ that is not placed in $\mathbb{N}^*$ in a trivial way?

Theorem (Dow (2014))

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- Dow used an Aronszajn tree in $2^{<\omega_1}$ to prove the existence of a so-called *nontrivial, maximal, nice* closed filter $\mathcal{F}$ on $\mathbb{N} \times 2^{\omega_1}$.

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    1. (An *Aronszajn tree* is a tree of uncountable height with no uncountable branches and no uncountable levels.)
    2. Here '<u>nice</u>' means that for every $F \in \mathcal{F}$, the set $\{n \in \mathbb{N} : F \cap (\{n\} \times 2^{\omega_1}) = \emptyset\}$ is finite.

$\exists n \quad \forall m \geq n$

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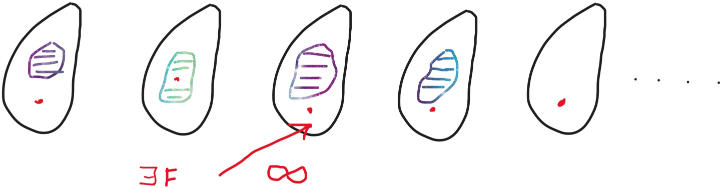

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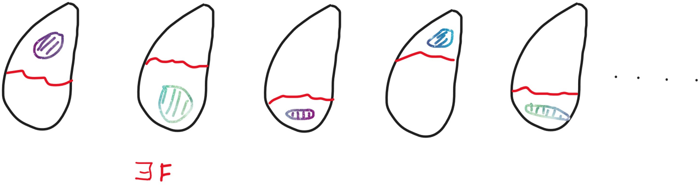

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- Each node of the Aronszajn tree is associated to a 'compatible' ultrafilter of regular open sets in some 2^{α}, for $\alpha < \omega_1$.
- This allowed Dow to do the same thing as above in $\beta(\mathbb{N} \times E(2^{\omega_1}))$ instead of $\beta(\mathbb{N} \times 2^{\omega_1})$.

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- So instead of in 2^{ω_1}, Dow used $E(2^{\omega_1})$, the projective cover (or absolute) of 2^{ω_1}. It is an extremally disconnected compact separable space of weight $\mathfrak{c}$.
- Each node of the Aronszajn tree is associated to a 'compatible' ultrafilter of regular open sets in some 2^{α}, for $\alpha < \omega_1$.
- This allowed Dow to do the same thing as above in $\beta(\mathbb{N} \times E(2^{\omega_1}))$ instead of $\beta(\mathbb{N} \times 2^{\omega_1})$.
- Kunen's machinery of constructing a weak P-point in $\mathbb{N}^*$ is used to embed $\beta(\mathbb{N} \times E(2^{\omega_1}))$ as a *weak P-set* in $\mathbb{N}^*$.

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There is copy of $\mathbb{N}^*$ in $\mathbb{N}^*$ that is a nowhere dense weak P-set.

- Instead of $E(2^{\omega_1})$ we use the Stone space of the measure algebra $\mathcal{M}_{\omega_1}$ on 2^{ω_1}.

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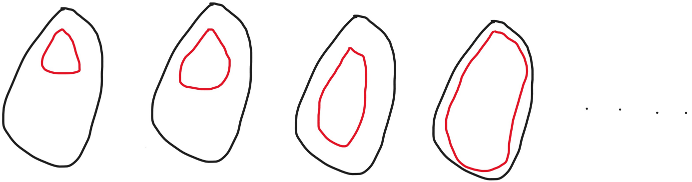
- It is known by the work of van Douwen and Chae and Smith that any nonspeudocompact space of countable π-weight has a remote point.
- We cannot apply that result, but in the case of measure algebras there is an easy way out.

- To see this, let X be any compact space, λ a Radon probability measure on X, with the property that $\lambda(A) = 0$ for any nowhere dense $A \subseteq X$. We claim that $\mathbb{N} \times X$ has a remote point.



$\lambda(F_1) \geq 1-2^{-1}$ . . . . $\lambda(F_3) \geq 1-2^{-3}$ . . . .

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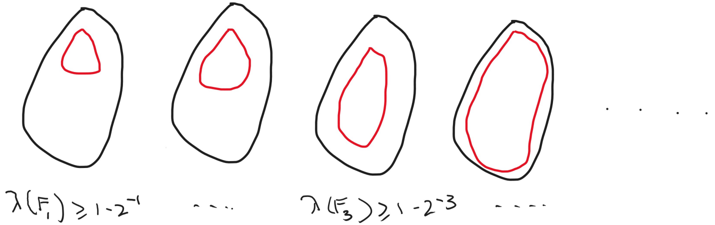

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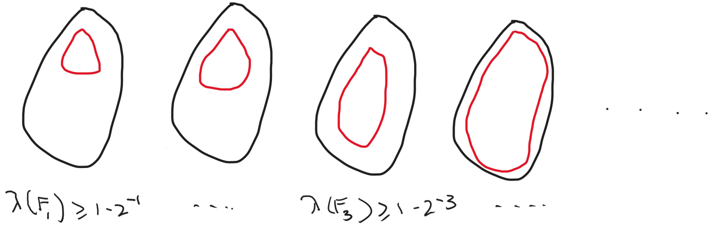

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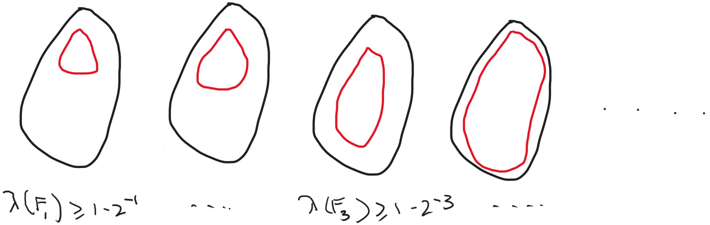

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- Now we do! *Never forget a good result!*
- These are the main ingredients for the (quite involved) proof of the theorem.

Theorem (Dow and vM (2020))

There is a copy X of $\mathbb{N}^*$ in $\mathbb{N}^*$ having the following properties:

1. *There is a countable subset E contained in $\mathbb{N}^* \setminus X$ such that the closure of E contains X,*
2. *for every countable discrete subset F in $\mathbb{N}^* \setminus X$, the closure of F misses X.*

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- We invite comments, corrections, more problems, …

THANK YOU!