Selective properties of products of Fréchet-Urysohn spaces

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Motivation

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Question

Is MA sufficient in the theorem above?

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"No" to the second one, the first one is open.

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- At stage α we face a pair $\langle S_\alpha, x_\alpha \rangle$, and have to do something if x_α is a limit point of S_α with respect to τ_α.

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- No problem to select Y_α since τ_α has weight $< \mathfrak{c} = \mathfrak{p}$ by recursive assumption, and hence $\langle \omega, \sigma_\alpha \rangle$ is FU.

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Corollary (BMZ 202?)

MA is consistent with the existence of two FU spaces with non-M-separable product.

One more result on products

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It improved the following earlier result.

Theorem (Repovš-Z. 2016)

In the Laver model, the product of any two H-separable countable spaces is M-separable. □

More questions

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Is it consistent that the product of two (m)H-separable countable spaces is (m)H-separable? Is it consistent that the product of two countable FU spaces is (m)H-separable? What happens in the Laver model? □

Let us note that by a result of Dow (2014) there are countable regular FU spaces with π-weight at least $\mathfrak{b}$, i.e., non-trivial ones in the context of the question above.

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Does PFA imply that the product of two FU spaces is (m)H-separable? R-separable? Does PFA imply that the product of two γ-spaces is Hurewicz?

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- *There exists a countable FU (and hence also mH-separable) space which is not H-separable;*

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Glory to >40 Nations helping Ukraine to survive!