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# *ZFC solution to 9 problems of Tkachuk on functional countability*

Functional countability has to do with real-valued continuous functions, so "space" will mean "Tychonoff space."

**Definition 1.1.** A *functionally countable space* is a space $X$ such that every continuous $f : X \to \mathbb{R}$ has countable range.

As is well known, a compact space is functionally countable if, and only if, it is scattered. The compact spaces we will be looking at have very simple scattered structure. They are the one-point compactifications $X + 1$ of spaces $X$ whose nonisolated points constitute a closed discrete subspace.

Our spaces $X$ are thus of Cantor-Bendixson rank 2, with the isolated points on level 0 and the non-isolated points on level 1. Of course, $X$+1 is of rank 3, with the sole point of level 2 being the extra point.

At SUMTOPO 2022 in Vienna last week, several speakers talked about separable compact spaces of this form, and their Banach spaces $C(X+1)$. This already gives a varied assortment of Banach spaces.

Vladimir Tkachuk ended a paper on functionally countable spaces with a list of 14 open questions about functional countability for spaces of the form $X^2 \setminus \Delta_X$. The only one that asked for a ZFC example was:

**QUESTION 4.10.** *Is there a ZFC example of a non-metrizable compact space $X$ such that $(X \times X) \setminus \Delta_X$ is functionally countable?*

In the examples I give, I use $X$+1 for the compact space rather than $X$ as stated in Question 4.10 and the other questions posed by Tkachuk.

At Sumtopo 2022 in Vienna, I gave a class of examples that answer Question 4.10 in the affirmative:

**Main Vienna Theorem.** *Let $X$ be a space with a countable dense set $Q$ of isolated points, such that $X \setminus Q$ is a closed discrete subspace, and each point of $X \setminus Q$ has a (countable) compact neighborhood.*
*Then $(X+1)^2 \setminus \Delta_{X+1}$ is functionally countable.*

Instead of proving this theorem in the usual way, I showed it for a specific, down to earth example; the general theorem has essentially the same proof.

It is trivial to show that $X$+1 itself is functionally countable:
the function values on $X \setminus Q$ must converge to $f(\infty)$.
In particular, all but countably many must agree with $f(\infty)$.

The following example is chosen for easy visualizability. $X^2$ has the open unit square as the underlying set, while $(X+1)^2$ can be thought of as $(0,1]^2$ with a very different topology.

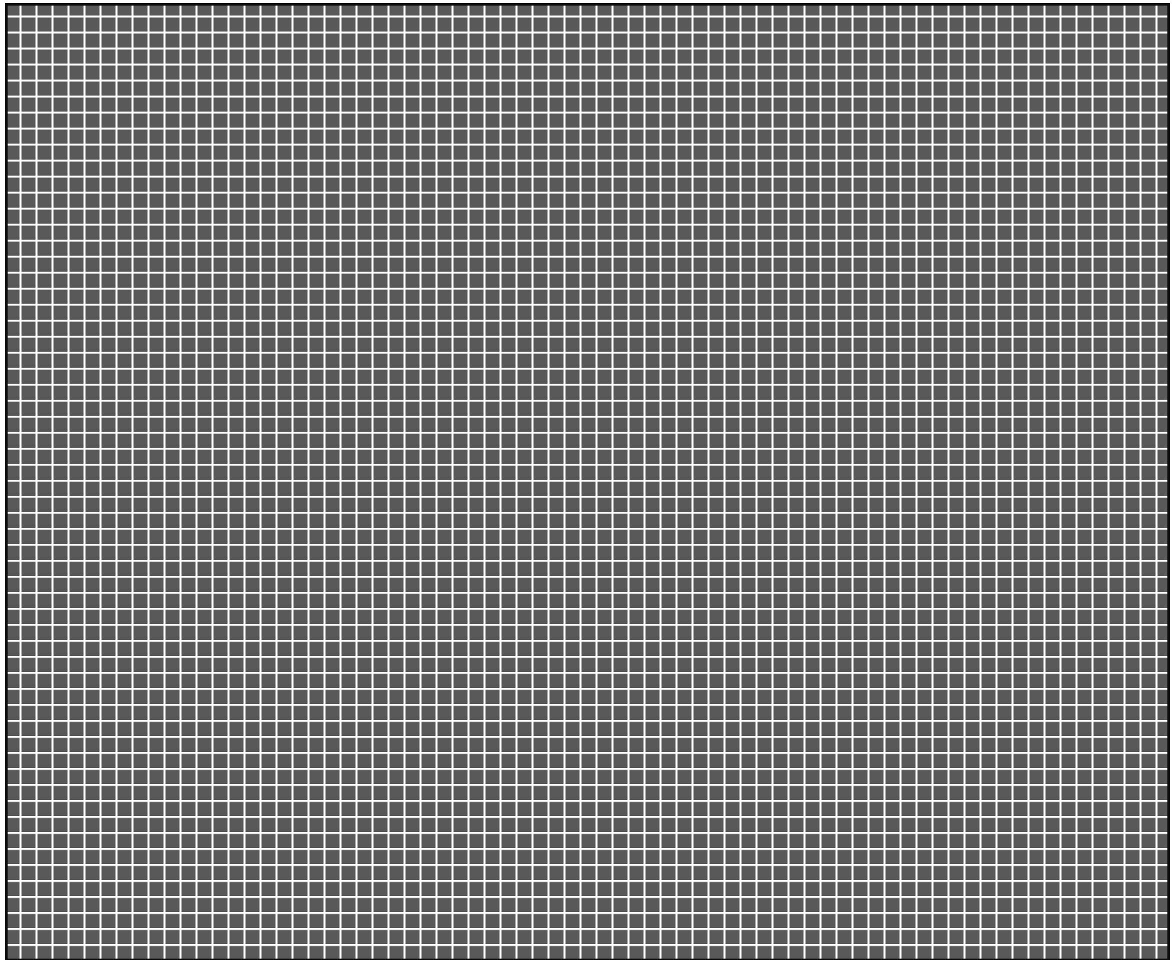
**Example 2.1.** *Let $X$ be the open unit interval $(0,1)$, with the following topology. Let $Q = \mathbb{Q} \cap (0,1)$ be a dense set of isolated points, and let each $p \in X \setminus Q$ have a base of neighborhoods consisting of $p$ together with the tails of an ascending sequence of points $\sigma_p(n)$ of $Q (n \in \omega)$.*

The *X* of Example 2.1 has a locally compact, locally countable topology that is much finer than the usual (Euclidean)topology on (0,1).

Vertical white lines represent $Q \times (X + 1)$ and horizontal lines represent $(X+1) \times Q$. The dark gray background represents $[(X+1) \setminus Q]^2$.

We will be summarizing the properties of continuous real-valued functions on $(X+1)^2$ minus the upper right corner point $(\infty, \infty)$.

But first, we look at Tkachuk's questions 4.1 through 4.8, which, put positively, ask for nonseparable examples. Example 2.1 leans heavily on the existence of the countable dense subset *Q*.



## 3. One answer for the first 8 questions

Tkachuk's first 6 questions are all answered by any counterexample for the first. The 7th and 8th are successively more demanding, but even the 8th is solved by the space described in the following slides, which is Fréchet-Urysohn.

**QUESTION 4.1.** *Suppose that $X$ is a compact space such that $(X \times X) \setminus \Delta_X$ is functionally countable. Must $X$ be separable?*

**QUESTION 4.2.** *Assume that $X$ is a countably compact space and $(X \times X) \setminus \Delta_X$ is functionally countable. Must $X$ be separable?*

**QUESTION 4.3.** *Assume that $X$ is a pseudocompact space and $(X \times X) \setminus \Delta_X$ is functionally countable. Must $X$ be separable?*

**QUESTION 4.4.** *Suppose that $X$ is a $\sigma$-compact space such that $(X \times X) \setminus \Delta_X$ is functionally countable. Is it true that $c(X) \leq \omega$?*

**QUESTION 4.5.** *Suppose that $X$ is a Lindelöf $\Sigma$-space such that $(X \times X) \setminus \Delta_X$ is functionally countable. Is it true that $c(X) \leq \omega$?*

**QUESTION 4.6.** *Suppose that $X$ is a Lindelöf space such that $(X \times X) \setminus \Delta_X$ is functionally countable. Is it true that $c(X) \leq \omega$?*

Every functionally countable compact space is scattered, and so its cellularity is equal to the number of isolated points. This is also its density.

**QUESTION 4.7.** *Suppose that $X$ is a compact space of countable tightness such that $(X \times X) \setminus \Delta_X$ is functionally countable. Must $X$ be separable?*

**QUESTION 4.8.** *Suppose that $X$ is a compact Fréchet–Urysohn space such that $(X \times X) \setminus \Delta_X$ is functionally countable. Must $X$ be separable?*

Recall that a *Bernstein subset* of a metric space $M$ is one that meets each uncountable closed subset of $M$, as does its complement. They exist in every complete *crowded* separable metric space via a transfinite induction. [A *crowded* space is one that has no isolated points.] This uses the fact that every uncountable closed set in a complete criowded metric space contains a copy of the Cantor set.

The following example answers Questions 1 through 8.

**Example 3.1.** Divide $(0,1)$ into disjoint Bernstein sets $B_0$ and $B_1$. The space $X$ has underlying set $(0,1)$ and topology $\tau$ in which $B_0$ is a dense set of isolated points, and each point $p$ of $B_1$ has a sequence in $B_0$ converging to it in $\mathcal{E}$, the Euclidean topology. With this sequence, we associate a base of clopen sets for $p$ as in Example 2.1.

We need to choose the sequences carefully. The method used owes a lot to a technique pioneered by Eric van Douwen. It uses a transfinite induction which, in the way used here, ensures that every Cantor set in $(0,1)$ has $\mathfrak{c}$-many points in the $\tau$-closure of $B_0$.

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Let $\prec$ be a strict well-order on $B_1$.Let $\langle D_\alpha : \alpha < \mathfrak{c}\rangle$ list the countable subsets of $B_0$ with uncountable closure, with each subset listed $2^{\aleph_0} = \mathfrak{c}$-many times. Note that the $\mathcal{E}$-closure of each $D_\alpha$ has cardinality $\mathfrak{c}$, and that every countable $\mathcal{E}$ crowded subset is among the $D_\alpha$. At the $\alpha$th stage of the induction, assume that $p_\nu \in B_1$ and $\sigma_\nu$ have been defined for all $\nu < \alpha$. [In the base case $\alpha = 0$, this is vacuously true.] If $\alpha < \mathfrak{c}$, there exists $p \in B_1$ in the $\mathcal{E}$-closure of $D_\alpha$ which is not among the earlier $p_\nu$; let $p_\alpha$ be the $\prec$-least such $p$, and use a sequence $\sigma_\alpha$ in $D_\alpha$ that $\mathcal{E}$-converges to $p_\alpha$ to define the $\tau$-neighborhoods of $p_\alpha$ as in Example 2.1.

An alternative, which may be simpler conceptually for some, is to let the $D_\alpha$ list the countable crowded subsets of $B_0$, since each has a Cantor set in its Euclidean closure.

This completes the ***description*** of Example 3.1. The proof that it answers Tkachuk's questions 4.1 through 4.8 takes some work.

When the induction in Example 3.1 is complete, every point in $B_1$ has been given a base of countable, compact neighborhoods. The issue of functional countability is much more complicated than in the separable case. For one thing, every point of $B_0^2$ is isolated, so in this subspace there is complete freedom to define continuous functions, and we have to find the right $D_\alpha$ to give us a contradiction for each uncountable subset of $B_0^2 \setminus \Delta_X$.

To see what these contradictions might be, we return to Example 2.1 to see how the continuous functions for it have to behave.

**Example 2.1.** *Let $X$ be the open unit interval $(0,1)$, with the following topology. Let $Q = \mathbb{Q} \cap (0,1)$ be a dense set of isolated points, and let each $p \in X \setminus Q$ have a base of neighborhoods consisting of $p$ together with the tails of an ascending sequence of points $\sigma_p(n)$ of $Q (n \in \omega)$.*

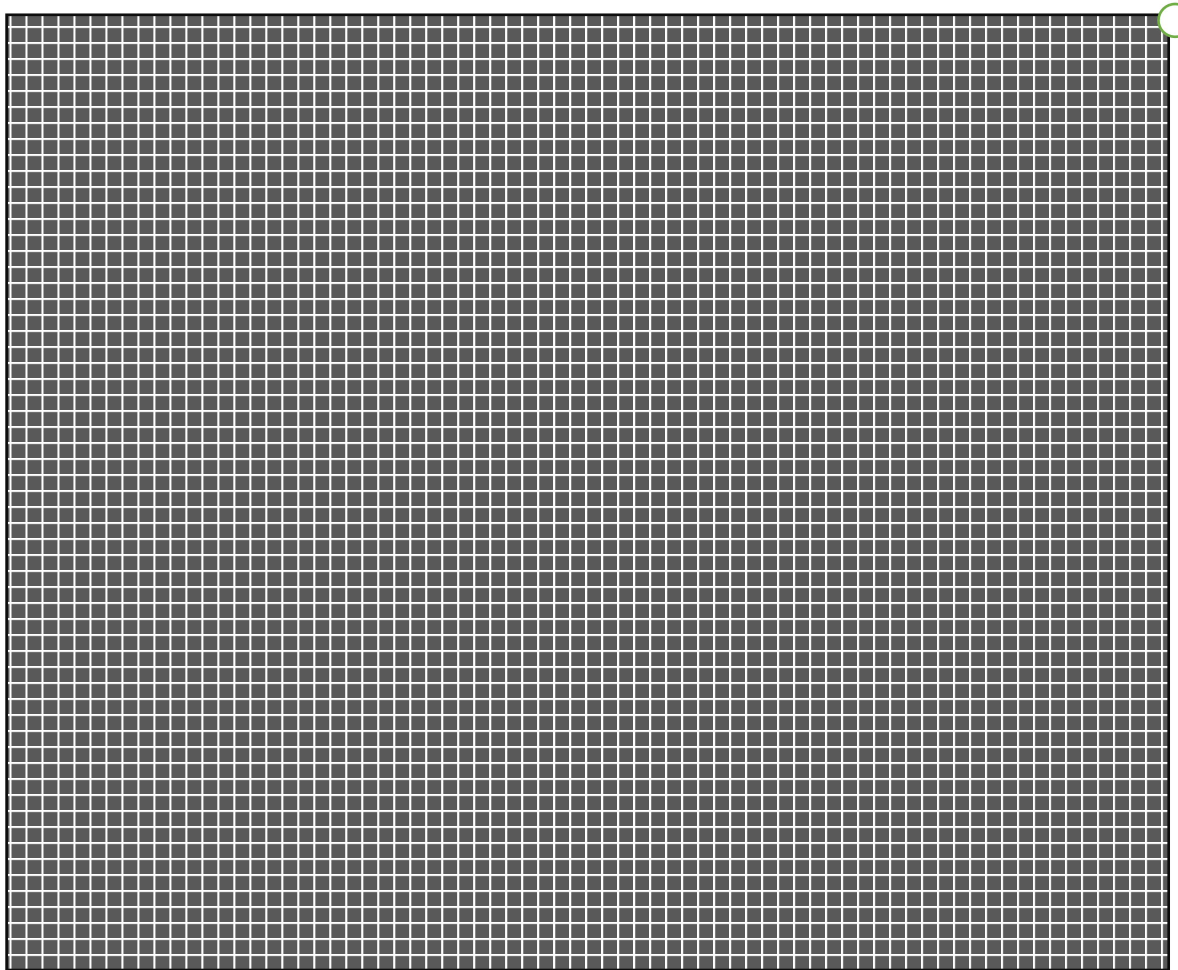
As mentioned earlier, $X + 1 = X \cup \{\infty\}$ is the one-point compactification of $X$.

The proof will show that $Y = (X + 1)^2 \setminus \{(\infty, \infty)\}$ is functionally countable. The presence or absence of $\Delta_X$ has no real effect on the proof.

The proof makes heavy use of the fact that each column $\{x\}$ x ($X$ + 1) and each row ($X$ + 1) x $\{y\}$ is homeomorphic to $X$ + 1.

Fix a continuous $f$: $Y \rightarrow \mathbb{R}$.

The restriction of $f$ to each row agrees with $f(\infty , y)$ for all but countably many $(x , y)$ in the row and its restriction to each column agrees with $f(x , \infty)$ for all but countably many points in the column.



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Step 1: Systematizing the columns of $Q \times [X+1]$

For each column $\{q\} \times [X+1]$, let $B_q = \{y \in X : f(q,y) \neq f(q,\infty)\}$. Each $B_q$ is countable, hence so is $B = \bigcup \{B_q : q \in Q\}$.

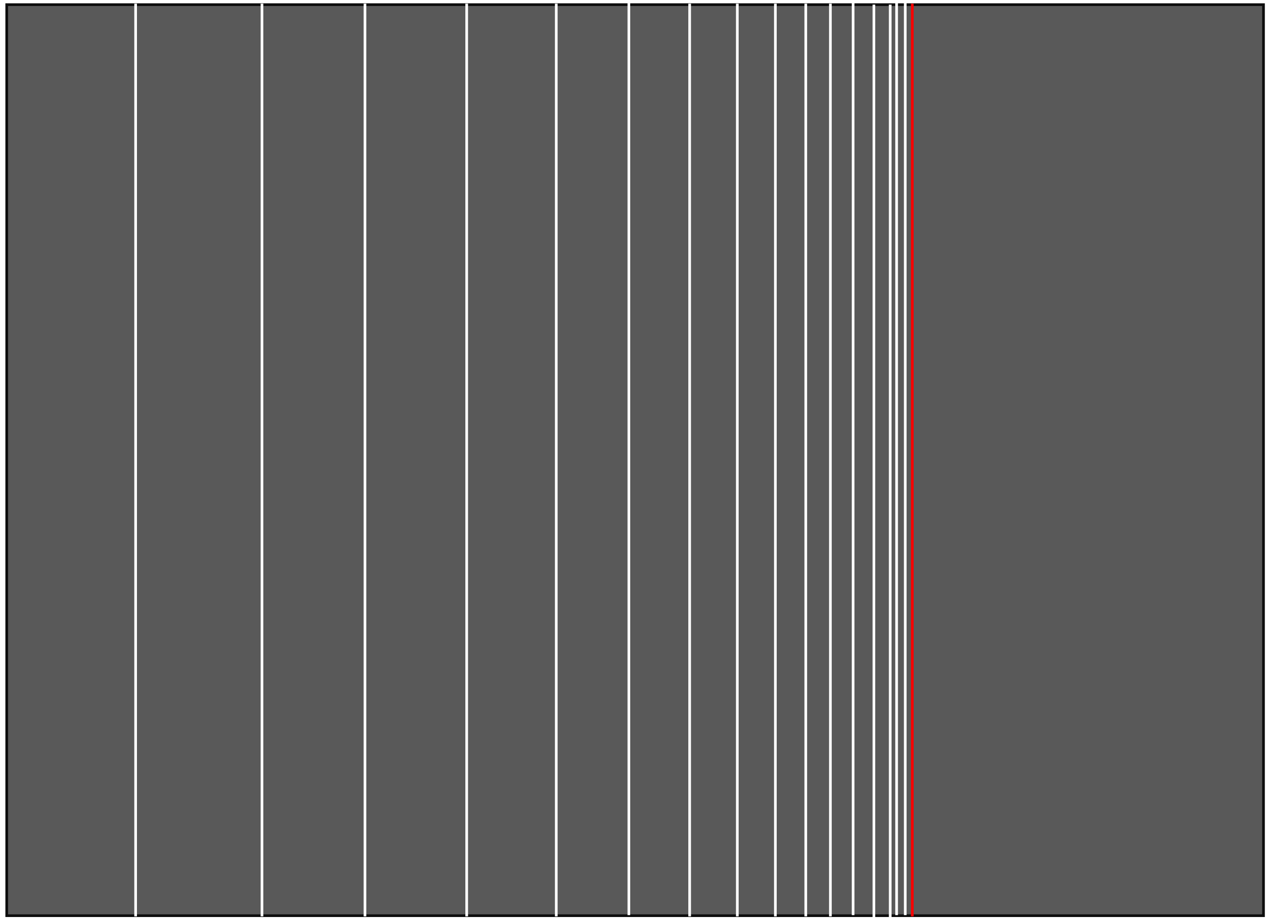
Let $Y_0 = Y \setminus ([X+1] \times B)$. The restriction of $f$ to each $Q$-column of $Y_0$ is constant, but no two of these constants need be equal. After all, each $Q$-column is clopen in $[X+1]^2$.

We are removing the "nonconforming" points in Column $q$ to get $B_q$, and then cutting out all rows with a rational "nonconformist" in them.

## Step 2: The effect of $D = Q \times ([X+1] \setminus B$ on the rest of $Y_0$

The rectangle below represents $Y_0$. This omits the countably many rows that make up $(X + 1) \times B$.

Because $D$ is dense in $Y_0$, and $f$ is continuous, and $f$ is constant on each column of $D$, it will be constant on every vertical line of $Y_0$ .

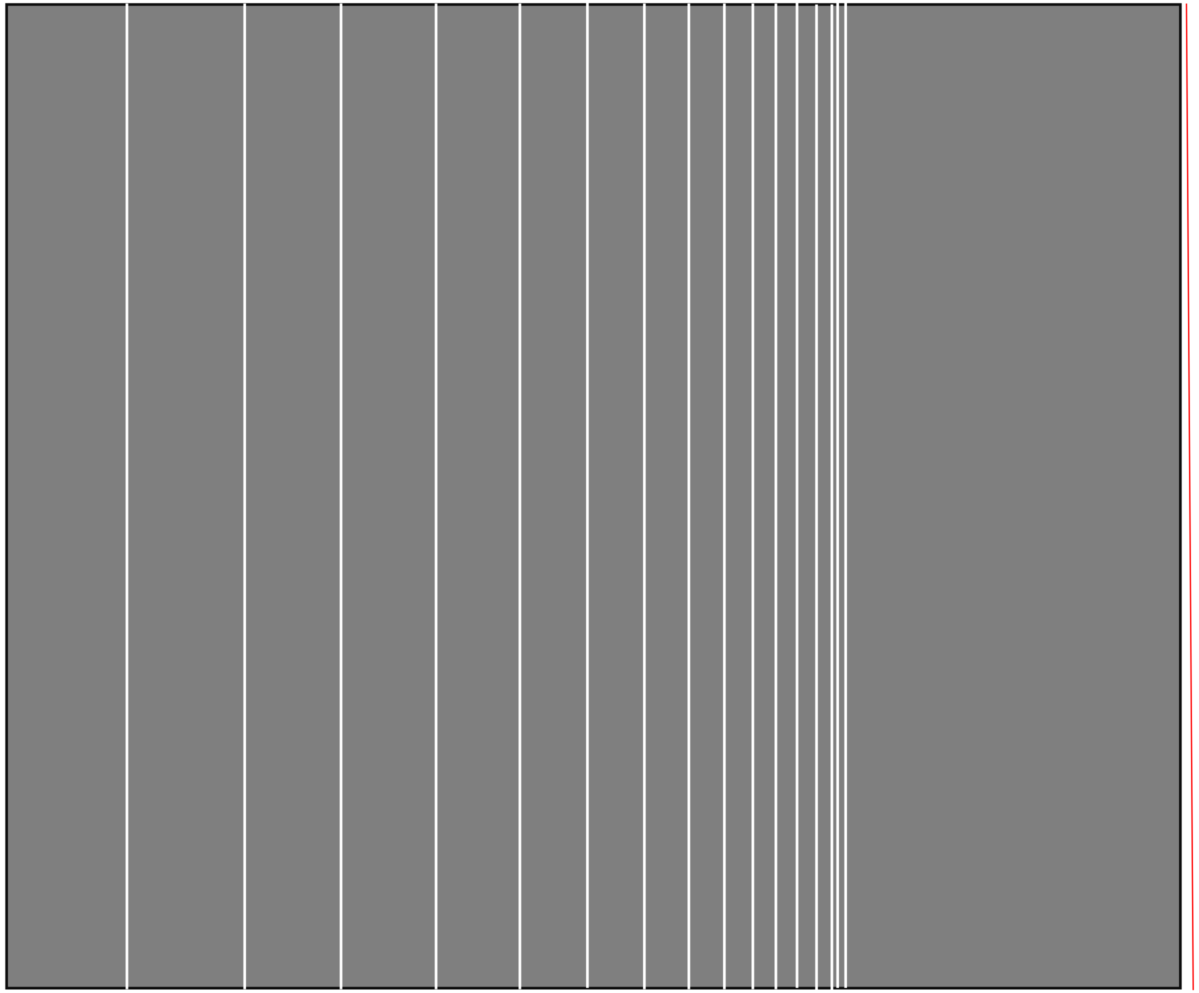


The vertical red line represents $\{p\} \times ([X+1] \setminus B)$, where $p \in X \setminus Q$.

The vertical white lines represent the sequence $\langle \{q_n\} \times ([X+1] \setminus B) \rangle$ where the $q_n$ τ-converge to $p$.

On the other hand, when $p = \infty$, almost all sequences converge to p, including all sequences from Q that converge to some point x in the Euclidean topology but are disjoint from the canonical sequence that converge to x in the finer topology.

The figure to the right deals with the case $p = \infty$. The columns converge to Column $x$ (not labeled) of $Y_0$ in the Euclidean topology, but in the topology on $X$+1, they converge to Column $\infty$ (orange). So $f$ is constant on the whole orange column.



And then all but countably many vertical lines of $Y_0$ must agree with the same constant $C$. And those which do not, must have their function values converge to $C$.

The result is that the restriction of $f$ to $Y_0$ is the composition of the projection $\pi$ to the first coordinate with a continuous function $g$ on $X$+1. Functional countability of $f$ follows quickly.

We now return to the main example:

**Example 3.1.** Divide $(0,1)$ into disjoint Bernstein sets $B_0$ and $B_1$. The space $X$ has underlying set $(0,1)$ and topology $\tau$ in which $B_0$ is a dense set of isolated points, and each point $p$ of $B_1$ has a sequence in $B_0$ converging to it in $\mathcal{E}$, the Euclidean topology. With this sequence, we associate a base of clopen sets for $p$ as in Example 2.1.

In each of the following four cases, $A$ stands for an uncountable subset of $X^2 \setminus \Delta_X$ such that the restriction of a real-valued function $f$ to $A$ has uncountable range. In each case, there will be a way to show that $f$ is not continuous everywhere on $X^2 \setminus \Delta_X$.

Case I. $A \subset B_0^2$.

Case II. $A \subset B_0 \times B_1$.

Case III. $A \subset B_1 \times B_0$.

Case IV. $A \subset B_1^2$.

In every case, $A$ can be chosen to have $f \restriction_A$ be injective. Also, since $A$ cannot have uncountably many elements in any row or any column, we can assume $A$ meets each row and each column in at most one point.

In all four cases, we take advantage of the fact that each $(x,y) \in A$ can be recovered from either of its coordinates.

There is an obvious symmetry between Case II and Case III, but Case I and Case IV pose different problems than they do, and also than each other.

In Case I and Case II, the first step is to find a suitable denumerable subset $\{p_n = (x_n, y_n) : n \in \omega\}$ of $A$ so that $\{f(p_n) : n \in \omega\}$ has a crowded subset, and such that $\{x_n : n \in \omega\}$ has uncountable $E$-closure and, therefore, uncountable $\tau$-closure.

It is easy to find crowded $\{f(p_n) : n \in \omega\}$ because we are working in $\mathbb{R}$. However, it could be true that $\{x_n : n \in \omega\}$ has countable $E$-closure. If so, let $\{x_n : n \in \omega\} = X_0$ and let $\{f(p_n) : n \in \omega\} = Z_0$. If $Z_\nu$ and $X_\nu$ have been defined for all $\nu < \alpha$ such that $X_\nu$ has countable closure, let $Z_\alpha$ be a crowded subset of $f[A] \setminus \bigcup\{Z_\nu : \nu < \alpha\}$, let $P_\alpha = f^{-1}Z_\alpha$ and let

$$X_\alpha = \pi_1[P_\alpha] \cup \bigcup\{X_\nu : \nu < \alpha\}.$$

Since the $X_\alpha$ are strictly $\subset$-increasing, and every uncountable subset of $\mathbb{R}$ has a countable crowded subset, there has to exist $\alpha < \omega_1$ such that $X_\alpha$ has uncountable $\tau$-closure. Let $\{x_\mu : \mu < \mathfrak{c}\}$ list all the points of this $\tau$-closure outside $X_\alpha$; they will all be in $B_1$.

Correction: $P_\alpha$ should stand for the intersection of the preimage of $Z_\alpha$ with $A$.

Since the $X_\alpha$ are strictly $\subset$-increasing, and every uncountable subset of $\mathbb{R}$ has a countable crowded subset, there has to exist $\alpha < \omega_1$ such that $X_\alpha$ has uncountable $\tau$-closure. Let $\{x_\mu : \mu < \mathfrak{c}\}$ list all the points of this $\tau$-closure outside $X_\alpha$; they will all be in $B_1$.

A key fact about $X_\alpha$ is that it is a countable set of isolated points whose $\tau$-closure provides the first coordinates of points that we will be using to get a contradiction.

Case II, $A \subset B_0 \times B_1$ now follows routinely:

$\vdash$ For each $x_\mu$, let $\langle x_n^\mu : n \in \omega\rangle \longrightarrow_\tau x_\mu$, where $x_n^\mu \in X_\alpha$ for all $n$.

$X_\alpha$ is the set of first coordinates of uniquely determined points in $P_\alpha$ .

For each $\mu$, let $p_n^\mu = (x_n^\mu, y_n^\mu)$. As noted above, $x_n^\mu$ uniquely determines $p_n^\mu$. Now the points $y_n^\mu$ of $B_1$ are all distinct and so they $\tau$-converge to $\infty$. Therefore, $p_n^\mu \longrightarrow_\tau (x_\mu, \infty)$.

Now, if $f$ is continuous, $\langle f(p_n^\mu)\rangle$ must converge to $f(x_\mu, \infty)$. We have limited control over which $x_\mu$ are in the $\tau$-closure of $X_\alpha$, but we can find disjoint closed intervals $[a, b]$ and $[c, d]$ such that $f[P_\alpha)] \cap [a, b])$ and $f[P_\alpha] \cap [c, d])$ have disjoint, uncountable closures, because $f(P_\alpha)$ has $\mathfrak{c}$-many condensation points.

This is very different than the behavior that a continuous function on a separable subspace must have, and so continuity of *f* is contradicted.

Case I, $A \subset B_0^2$, has a proof that has much in common with the proof in Case II, but it is complicated by the way the points $y_n$ are in $B_0$ rather than in $B_1$.

$\vdash$ It still could be the case that $y_n^\mu \longrightarrow_\tau \infty$ for uncountably many $\mu$, reducing the argument to the one for Case II. This happens, for example, whenever $y_n^\mu \longrightarrow_\mathcal{E} y \in B_0$.

This leaves the case where there is a subsequence of $\langle y_n^\mu : n \in \omega\rangle$ which $\tau$-converges to a point $y_\mu$ of $B_1$ for all but countably many $\mu$. In this case, we have uncountably many $p_\mu$ in the $\tau$-closure of $P_\alpha$, and this is essentially the case covered in Example 2.1 $\dashv$

It remains to take care of:

Case IV. $A \subset B_1^2$. This is proven much like Case II, but since $A$ is closed discrete, the only way to get a contradiction is to extend $f\restriction_A$ to put enough of $A$ into the closure of a countable set of points of $B_0 \times B_1$ that works somewhat like $X_\alpha$ did there.

$\vdash$ Follow the construction of $X_\alpha$ , but interpret all limits and closures as $\mathcal{E}$-limits and $\mathcal{E}$-closures, and disregard the sentence where $\{x_\mu : \mu < \mathfrak{c}\}$ is defined. Let $\langle p_k = (x_k, y_k) : k \in \omega\rangle$ be a one-to-one listing of $X_\alpha$. For each $p_k$, find $\langle x_n^k : n \in \omega\rangle \longrightarrow_\tau x_k\rangle$, $x_n^k \in B_0\ \forall n, k$. Then $\{x_n^k : n \in \omega, k \in \omega\}$ is the countable set of isolated points that does the job that $X_\alpha$ did in Case II. It has uncountable $\mathcal{E}$-closure, hence uncountable $\tau$-closure.

The remaining 5 questions ask for much more structure on the spaces involved, and we do not even have consistency results for them.

**QUESTION 4.9.** *Suppose that $X$ is a compact $\omega$-monolithic space such that the space $(X \times X) \setminus \Delta_X$ is functionally countable. Must $X$ be countable?*

**QUESTION 4.11.** *Suppose that $X$ is a linearly orderable space and $(X \times X) \setminus \Delta_X$ is functionally countable. Must $X$ be separable?*

**QUESTION 4.12.** *Suppose that $X$ is a Lindelöf $P$-space such that $(X \times X) \setminus \Delta_X$ is functionally countable. Must $X$ be countable?*

**QUESTION 4.13.** *Assume that $(X \times X) \setminus \Delta_X$ is functionally countable and $X$ is monotonically normal. Must $X$ be separable?*

**QUESTION 4.14.** *Suppose that $X$ is a monotonically normal compact space such that $(X \times X) \setminus \Delta_X$ is functionally countable. Must $X$ be separable?*

There is a natural strengthening of the 9 solved problems that gives a very different picture. It is to ask that every **closed** subspace of $X^2 \setminus \Delta_X$ be functionally countable. Then even Question 10 becomes very open, and the only example I have for this modification of Questions 1 through 8 uses the axiom $\diamondsuit$. If this modification of the above questions turns out to have negative answers in ZFC, the argument might then give ideas for negative answers in ZFC to the above questions themselves.