$\Box, \nabla + D$

Paul Gartside joint work with Hector Barriga-Acosta

TOPOSYM — July, 2022

What are *D*-spaces?
What are they good for?
Key open problems?

D-Space Definition

Let $X = (X, \tau)$ be a (Tychonoff) space.

Definition

A neighbornet is a map $U : X \to \tau$ such that

x is in $U(x)$ for every x.

Definition

A space X is D if for every neighbornet U there is a closed discrete subset E such that $U(E) := \bigcup\{U(x) : x \in E\} = X$.

Compact $\Rightarrow D$ metrizable $\Rightarrow D$ ω_1 is not D

What are *D*-spaces Good For?

Theorem (Grothendieck)

Let X be compact.
Then every countably compact subset of $C_p(X)$ is compact.

$C_p(X) =$ all cts $f : X \to \mathbb{R}$, pointwise topology

countably compact $\iff$ closed discrete subsets are finite

Theorem (Buzyakova)

If X compact then $C_p(X)$ is hereditarily D.

Lemma

Let Y be a D-space. Then:

Y countably compact $\Rightarrow$ compact & $L(Y) = e(Y)$.

countably compact $\iff$ closed discrete subsets are finite

$e(Y) \leq \kappa \iff$ every closed discrete subset has size $\leq \kappa$
$L(Y) \leq \kappa \iff$ every open cover has a subcover of size $\leq \kappa$

Lemma

Let Y be a D-space. Then:

Y countably compact $\Rightarrow$ compact & $L(Y) = e(Y)$.

Theorem (Gruenhage)

If X Lindelöf Σ then $C_p(X)$ is hereditarily D.

What Else do we Know about ***D***?

There are various results of the type:

if X has THIS *structure then X is D*.

And also:

Theorem

Let X be a D-space. Then X is irreducible:

every open cover has a minimal open refinement.

Daniel Soukup $\exists X$ not D, but all closed subsets irreducible.

Built using Shelah's club guessing.

Basic *D*-space Open Problem

But *nothing* in the converse direction:

Question

Is X a D-space if:

Lindelöf, *or* *hereditarily Lindelöf,*

or paracompact, *or* *meta-Lindelöf?*

Are paracompact spaces D?

What are Box and Nabla Products?

When are they normal or paracompact?

THE Box Product Problem

Definitions

Let $(X_n)_n$ be spaces.

The box product, $\square_n X_n$, is $\prod_n X_n$ with topology generated by open boxes: $\prod_n U_n \quad (\forall n\ U_n$ open).

Write $\square X^\omega$ for $\square_n X_n$ where $\forall n\ X_n = X$.

Let $=^*$ be mod finite equivalence on $\prod_n X_n$.

The nabla product, $\nabla_n X_n$, is $(\square_n X_n) \,/ =^*$.

Write ∇X^ω for $\nabla_n X_n$ where $\forall n\ X_n = X$.

Paracompact or Normal?

Question

When are box or nabla products paracompact? normal?

Question (The Box Product Problem (1940s))

Is $\square[0,1]^\omega$ *paracompact?*

Answers and Observations

$\square = \square_n X_n$ and $\nabla = \nabla_n X_n$.

- ★ Box and nabla products are typically *not* paracompact if at least one factor is not compact
- ★ (ZFC) If all factors are compact but 'large' then $\square$ and ∇ need not be paracompact
- ★ When all factors are compact the quotient map $\square$ to ∇ is closed with σ-compact fibres

  Hence $\square$ is paracompact iff ∇ is paracompact
- ★ Consistently box and nabla products of compact metrizable spaces are paracompact
- ★ (CH) Box and nabla products of compact scattered spaces are paracompact

Answers and Observations

$\square = \square_n X_n$ and $\nabla = \nabla_n X_n$.

- Box and nabla products are typically *not* paracompact if at least one factor is not compact
- (ZFC) If all factors are compact but 'large' then $\square$ and ∇ need not be paracompact
- When all factors are compact the quotient map $\square$ to ∇ is closed with σ-compact fibres

  Hence $\square$ is paracompact iff ∇ is paracompact
- Consistently box and nabla products of compact metrizable spaces are paracompact
- (CH) Box and nabla products of compact scattered spaces are paracompact

Answers and Observations

$\square = \square_n X_n$ and $\nabla = \nabla_n X_n$.

- ★ Box and nabla products are typically *not* paracompact if at least one factor is not compact
- ★ (ZFC) If all factors are compact but 'large' then $\square$ and ∇ need not be paracompact
- ★ When all factors are compact the quotient map $\square$ to ∇ is closed with σ-compact fibres

  Hence $\square$ is paracompact iff ∇ is paracompact
- ★ Consistently box and nabla products of compact metrizable spaces are paracompact
- ★ (CH) Box and nabla products of compact scattered spaces are paracompact

Answers and Observations

$\square = \square_n X_n$ and $\nabla = \nabla_n X_n$.

★ Box and nabla products are typically *not* paracompact if at least one factor is not compact

★ (ZFC) If all factors are compact but 'large' then $\square$ and ∇ need not be paracompact

★ When all factors are compact the quotient map $\square$ to ∇ is closed with σ-compact fibres

Hence $\square$ is paracompact iff ∇ is paracompact

★ Consistently box and nabla products of compact metrizable spaces are paracompact

★ (CH) Box and nabla products of compact scattered spaces are paracompact

Answers and Observations

$\square = \square_n X_n$ and $\nabla = \nabla_n X_n$.

- ★ Box and nabla products are typically *not* paracompact if at least one factor is not compact
- ★ (ZFC) If all factors are compact but 'large' then $\square$ and ∇ need not be paracompact
- ★ When all factors are compact the quotient map $\square$ to ∇ is closed with σ-compact fibres

  Hence $\square$ is paracompact iff ∇ is paracompact
- ★ Consistently box and nabla products of compact metrizable spaces are paracompact
- ★ (CH) Box and nabla products of compact scattered spaces are paracompact

We lack:

Consistent examples – to match the consistent theorems

ZFC theorems – to balance the ZFC examples

When are box or nabla products *D*-spaces?

Special interest:

theorems in ZFC, and

when we know $\square$ or ∇ is paracompact

Scattered spaces, $\square$ of compact not *D*, ‘small’ or ‘nice’ spaces

Summary - Scattered Spaces

Theorem

Let X be hereditarily paracompact and scattered.

Then $\square X^\omega$ and ∇X^ω are hereditarily D.

Theorem

Let X be scattered of finite height.

Then $\square X^\omega$ and ∇X^ω are hereditarily D.

Theorem

Let $(X_n)_n$ be scattered spaces with

bounded finite scattered height.

Then $\square_n X_n$ and $\nabla_n X_n$ are hereditarily D.

Example

There is a sequence $(X_n)_n$ of spaces where X_n is scattered of height n, but $\nabla_n X_n$ is not D.

Topological Partial Orders

A partial order, $\preceq$, on space X is topological
if down-sets, $\downarrow x := \{y : y \preceq x\}$, are open.

Proposition

Let X have a topological partial order $\preceq$.

(1) If every up-set, $\uparrow x$, is D then X is D.

(2) If every up-set is hereditarily D then X is hereditarily D.

Example

Let X be scattered. Define $x \preceq_{sc} y$ iff $x = y$ or $ht(x) < ht(y)$.
Then $\preceq_{sc}$ is topological.

Hereditarily Paracompact and Scattered

Lemma
For every hereditarily paracompact scattered space X there is a topological partial order $\preceq_X$ such that every up-set, $\uparrow x$, is finite

Theorem
Let X be hereditarily paracompact and scattered. Then $\square X^\omega$ and ∇X^ω are hereditarily D.

Consider the product partial order, $\preceq_X^\omega$.

Down-set is product of down-sets... hence open.

Up-set is product of up-sets... hence discrete, and so D.

Finite Scattered Height

Theorem

Let X be scattered of finite height.

Then $\Box X^\omega$ and ∇X^ω are hereditarily D

Consider the product partial order, $\preceq_{sc}^{\omega}$.
Proceed by induction on scattered height.

Liang-Xue Peng:

$\Box X^\omega$ is D if X is scattered of finite height.

Theorem

Let $(X_n)_n$ be scattered spaces

with bounded finite scattered height.

Then $\Box_n X_n$ and $\nabla_n X_n$ are hereditarily D.

Unbounded Finite Scattered Heights

Example

There is a sequence $(X_n)_n$ of spaces where X_n is scattered of height n, but $\nabla_n X_n$ is not D.

Lemma

If X is the increasing union of X_n's

then X_δ embeds as a closed set in $\nabla_n X_n$.

Example

There is a P-space X which is the increasing union of subspaces X_n where each X_n is scattered of height n, but X is not D.

The example is built from a 'base' space B.
$B = \{0,1\}^{\beth_{\omega_1}}$ with $< \beth_{\omega_1}$ boxes!

Box of Compact Not *D*

We have two examples of...

Example

There are compact X such that $\Box X^\omega$ (and ∇X^ω) is not D.

Specifically, $X = \{0,1\}^{\mathfrak{c}^+}$ works.

A modified version of our scattered example above
embeds as a closed set in ∇.

Summary - `Small' Spaces, Consistently

Theorem (Model Hypothesis)

The nabla product of
first countable spaces of size $\leq \mathfrak{c}$ is hereditarily D.

Hence, the box product of compact, first countable spaces is D.

The *Model Hypothesis*: For some κ, $H(\mathfrak{c})$ is the increasing union of H_α's, for $\alpha < \kappa$, where each H_α is an elementary submodel of $(H(\mathfrak{c}), \in)$ and each $H_\alpha \cap \omega^\omega$ is not $\leq^*$-cofinal.

Theorem ($\mathfrak{d} = \omega_1$)

(a) If X has weight $\leq \omega_1$ then ∇X^ω is hereditarily D.

(b) If X is compact and has weight $\leq \omega_1$ then $\Box X^\omega$ is D.

Summary - Metrizable Spaces or 1^o, in ZFC

Theorem

Let X be a metrizable space. Then ∇X^ω is hereditarily D.

If X has weight no more than $\mathfrak{d}$ then $\square X^\omega$ is hereditarily D.

Theorem

Let X be a first countable space with weight no more than $\mathfrak{d}$ and strictly less than $\aleph_\omega$. Then ∇X^ω is hereditarily D.

Conclusions and Reference

- ★ Box and nabla products of compacta need not be D
- ★ In all cases we (consistently) know a box or nabla product is paracompact we now know it is D (often hereditarily)
- ★ For a wide variety of spaces their box and nabla products are D (often hereditarily) in **ZFC**

Question

Is the box product of compact scattered spaces D?
Are nabla products of compact scattered spaces hereditarily D?
At least true under (CH)*?*

Box and Nabla Products that are D-Spaces
Hector A. Barriga-Acosta, Paul M. Gartside
arXiv:2111.10482 `https://arxiv.org/abs/2111.10482`