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Recurrence and Universal Minimal Recurrence

(B. Weiss)

X - compact metric space, $T: X \to X$ cont.

$x \in X$ is recurrent if $T^{n_i} x \to x$, $n_i \to \infty$.

More refined notions

1) uniform recurrence : $\{n : T^n x \in U_x\}$ is syndetic for all neighborhoods U_x of x. (syndetic = bounded gaps).

This is equivalent to the orbit closure of x being a minimal set.

Def: x is proximal to y if for some $n_i \to \infty$ $d(T^{n_i} x, T^{n_i} y) \to 0$.

x is a distal point if it is not proximal to any $y \neq x$.

Thm: Given a system (X, T) and $x \in X$ there exists a uniformly recurrent point y that is proximal to x.

(related to Hindman's thm that if $A \subset \mathbb{N}$ contains an IP-set (= finite sums of an infinite sequence $\{p_j\}$, i.e. $\sum_{j \in F} p_j$, F finite subset $\mathbb{N}$) and $A = A_1 \cup A_2$ then either A_1 or A_2 contains an IP-set).

Def: A subset $A \subset \mathbb{N}$ is called a central set if for some pair (x, y) in (X, T) x is proximal to y, y is uniformly recurrent and $A = \{n : T^n x \in V_y\}$ where V_y is some nbd of y.

$\mathcal{C}$ - central sets. $\mathcal{C}^*$ - sets that intersect every $A \in \mathcal{C}$.

$\mathcal{C}^*$ has the finite intersection property,

and hence can speak of $\mathcal{C}^*$-convergence and recurrence:

A point x is $\mathcal{C}^*$-recurrent if $\{n : T^n x \in U_x\} \in \mathcal{C}^*$

for any nbd. U_x of x.

<u>Thm</u>: (Furstenberg-W.) The following are equivalent:

(1) x is $\mathcal{C}^*$-recurrent

(2) x is a distal point in its orbit closure

(3) For any system (Z, S) and recurrent point $z \in Z$ the point (x, z) is recurrent in $(X \times Z, T \times S)$.

Erdös and Stone (1945) studied recurrence sets (not quite in this generality).

$A \subset \mathbb{N}$ is a <u>set of recurrence</u> if for every (minimal) system (Z, S) there is a point $z \in Z$ $\{n : T^n z \in V_z\} \cap A$ is infinite for all nbds V_z of z.

OPEN PROBLEM (I. Katznelson ...)

If A is a set of recurrence ~~of~~ for all equicontinuous (rotations of compact groups) systems is A a set of recurrence for all minimal systems. ??

Def: A subset $A \subset \mathbb{N}$ is a universal minimal recurrence set (UMR) if for every minimal point $z \in Z$, and every nbd. V_z of z the set

$$\{n : T^n z \in V_z\} \cap A \quad \text{is infinite.}$$

Def: A subset A is distally central (d-central) if there is a pair of proximal points (x, y) in (X, T) such that y is uniformly recurrent distal and for some nbd V_y of y

$$A = \{n : T^n x \in V_y\}.$$

Thm: (Glasner-W.) Every d-central set is UMR.

Proof: Let A be d-central, (x, y, V_y) defining A, in (X, T) and $Y \subset X$ the orbit closure of y in X. Given a minimal system (Z, S) and a point $z \in Z$ first note that it is known that the pair (y, z) is uniformly recurrent in $Y \times Z$. Since (x, y) is a proximal pair there is a sequence n_i such that

$$(T^{n_i}x, T^{n_i}y, S^{n_i}z) \longrightarrow (y, y, z') \text{ for some } z' \in Z.$$

Since the orbit closure of (y, z) is minimal there is a sequence k_j with $(T^{k_j}y, S^{k_j}z') \to (y, z)$

If V_z is a nbd. of z this implies that for inf. many pairs n_i, k_j, $(T^{k_j}T^{n_i}x, T^{k_j}T^{n_i}y, S^{k_j}S^{n_i}z) \in V_y \times V_y \times V_z$ and since $A = \{n : T^n x \in V_y\}$ this implies that

$A \cap \{n : S^n z \in V_z\}$ is infinite. $\square$

The main result (Thm IV) of Erdös and Stone was that for equicontinuous systems (X, T) there is a sequence n_i such that both T^{n_i} and $T^{n_i^2}$ converge uniformly to the identity. They ask "...what can be said about the sequence of integers n for which $f^n(x)$ is in a given neighborhood of x, f being recurrent at x. Thus, under the hypotheses of Thm IV, this sequence contains infinitely many squares, and in fact infinitely many kth powers."

The squares $\{n^2\}$ are a <u>recurrence sequence</u>. In fact they are a <u>Poincaré</u> sequence, i.e. for any measure preserving system $(X, \mathcal{B}, \mu, T)$ and set of positive measure B, for some n

$$\mu(B \cap T^{-n^2} B) > 0.$$

However the squares are not an UMR. This follows from a much more general result. We say that a subset $A \subset \mathbb{N}$ is a <u>small</u> set if the only minimal point in the orbit closure of 1_A under the shift in $\{0,1\}^{\mathbb{Z}}$ is the point all of whose coordinates are zero. If $\mathcal{A}$ is the norm closed algebra of bounded functions on $\mathbb{Z}$ containing all <u>minimal</u> functions (i.e. functions of the form $f(T^n x_0)$ where $f \in C(X)$, and (X,T) is minimal) then $A \subset \mathbb{Z}$ is an <u>interpolation</u> set for $\mathcal{A}$ if <u>any</u> <u>bounded function</u> on A can be extended to an element of $\mathcal{A}$.

Thm (Glasner-W.) A subset $A \subset \mathbb{Z}$ is an interpolation set for $\mathfrak{A}$, if and only if it is a small set.

In the proof of this theorem we actually constructed for any assignment of 0,1 to the elements of a small set A, ~~these~~ this assignment can be extended to a minimal point in $\{0,1\}^{\mathbb{Z}}$. If $A = \{a_0, a_1, \dots\}$ is a small set, we may assume that $a_0 = 0$, then assigning the value 1 to a_0 and 0 to all other a_i, we get a minimal point x in $\{0,1\}^{\mathbb{Z}}$ that cannot return to a nbd determined by the cylinder set $\{\omega \in \{0,1\}^{\mathbb{Z}} : \omega(0) = 1\}$.

Note that the combination of this result with the Erdös-Stone result shows that the UMR version of Katznelson's problem has a negative answer.

There are UMR sets that are not d-central. For this let me recall a notion introduced by Haddad and Ott.

Def: A point x in (X,T) is weakly product recurrent (WPR) if for every minimal point z in (Z,S) the pair (x,z) is a recurrent point in $X \times Z$.

They gave examples of (WPR)-points that are not PR (= product recurrent, same definition but for all recurrent points z). Recently Eli Glasner and I showed that there are minimal points that are WPR but not PR. In fact we showed that there are weakly mixing such minimal points.

(X, T) is <u>topologically transitive</u> if it has a dense orbit. it is <u>weakly mixing</u> if $(X \times X, T \times T)$ is topologically transitive. A weakly mixing minimal system has no distal points. There exist minimal weakly mixing system (X, T) such that for all pairs (x_1, x_2) that are not on the same orbit, the orbit closure of (x_1, x_2) in $X \times X$ is all of $X \times X$. In fact X can be taken in $\{0,1\}^{\mathbb{Z}}$, and then for any $\xi \in X$, $\xi(0)=1$, $A = \{n \in \mathbb{N} : \xi(n) = 1\}$ will be a UMR – but not d-central.