Star-P and Weakly Star-P properties

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Star-CS Star-CE and Star-CCC

Spaces are considered Hausdorff and with at least two points. If X is a topological space and $\mathcal{U}$ is a family of subsets of X, then the star of a subset $A \subset X$ with respect to $\mathcal{U}$ is the set

$$st(A, \mathcal{U}) = \bigcup \{U \in \mathcal{U} : U \cap A \neq \varnothing\}$$

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$$st(A, \mathcal{U}) = \bigcup \{U \in \mathcal{U} : U \cap A \neq \varnothing\}$$

Definition

Let P be a topological property. A space X is said to be star-P if whenever $\mathcal{U}$ is an open cover of X, there is a subspace $A \subset X$ with the property P, such that $st(A, \mathcal{U}) = X$. The set A will be called a star kernel of $\mathcal{U}$.

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Definition

The spread $s(X)$ of a space X is defined as

$$s(X) = \sup \{|A| : A \subset X \text{ is discrete}\}$$

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Star-CS Star-CE and Star-CCC

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In the following star-CS denote the star-(countable spread) property and star-CE denote the star-(countable extent) property.

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Some well-known implications:

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Some well-known implications:

- separable $\Rightarrow$ star-countable

Star-CS Star-CE and Star-CCC

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- Lindelöf $\Rightarrow$ countable extent $\Rightarrow$ star-countable

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- star-countable $\Rightarrow$ star-(σ-compact) $\Rightarrow$ star-Lindelöf

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- star-Lindelöf $\Rightarrow$ feebly Lindelöf

Definition

A topological space X is called feebly Lindelöf if every locally finite family of non-empty open sets in X is countable.

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Proposition

If X has an uncountable locally finite cellular family, then X is not a star-CE space.

Star-CS Star-CE and Star-CCC

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If X is star-CE then X is feebly Lindelöf.

star-Lindelöf $\Rightarrow$ star-CE $\Rightarrow$ feebly Lindelöf

Example

If $S = \{\alpha + 1 : \alpha < \omega_1\}$ and $X = (\omega_1 \times \omega) \cup (S \times \{\omega\})$ is considered as a subspace of $\omega_1 \times (\omega + 1)$, then X is a Tychonoff star-CE space which is not star-Lindelöf.

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star-CE $\not\Rightarrow$ star-Lindelöf

Another implications:

Star-CS Star-CE and Star-CCC

Another implications:

- Separable $\Rightarrow$ CCC $\Rightarrow$ star-CCC

Star-CS Star-CE and Star-CCC

Another implications:

- Separable $\Rightarrow$ CCC $\Rightarrow$ star-CCC
- star-countable $\Rightarrow$ star-CCC

Star-CS Star-CE and Star-CCC

Another implications:

- Separable $\Rightarrow$ CCC $\Rightarrow$ star-CCC
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Proposition

If X is star-CCC then X is feebly Lindelöf.

Star-CS Star-CE and Star-CCC

Example

Let $\mathcal{A}$ be a MAD family in ω with $|\mathcal{A}| = 2^\omega$, and let $X = \Psi(\mathcal{A})$ be the psi-space related to $\mathcal{A}$. Let $Y = \alpha D$ be the one-point compactification of the discrete space D of size 2^ω. Then the space $X \times Y$ is a Tychonoff star-Lindelöf space which is not star-CCC.

Star-CS Star-CE and Star-CCC

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Star-CS Star-CE and Star-CCC

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star-Lindelöf $\not\Rightarrow$ star-CCC

Example

Let $\mathcal{F}[\mathbb{R}]$ be the hyperspace of all the non-empty finite subsets of $\mathbb{R}$ endowed with the Pixley-Roy topology. Then $\mathcal{F}[\mathbb{R}]$ is a CCC Moore space wich is not star-CE.

Star-CS Star-CE and Star-CCC

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star-CCC $\not\Rightarrow$ star-CE

More implications:

Star-CS Star-CE and Star-CCC

More implications:

- countable spread $\Rightarrow$ star-CS

Star-CS Star-CE and Star-CCC

More implications:

- countable spread $\Rightarrow$ star-CS
- star-countable $\Rightarrow$ star-CS

Star-CS Star-CE and Star-CCC

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- countable spread $\Rightarrow$ star-CS
- star-countable $\Rightarrow$ star-CS
- star-CS $\Rightarrow$ star-CE

Star-CS Star-CE and Star-CCC

More implications:

- countable spread $\Rightarrow$ star-CS
- star-countable $\Rightarrow$ star-CS
- star-CS $\Rightarrow$ star-CE
- star-CS $\Rightarrow$ star-CCC

Theorem (Šapirovskii)

If $s(X) \leq \kappa$ then there is a dense subspace $Y \subset X$ with $hL(Y) \leq \kappa$.

Star-CS Star-CE and Star-CCC

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A space X is star-CS if and only if X is star-(hereditarily Lindelöf).

Star-CS Star-CE and Star-CCC

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star-CS $\Rightarrow$ star-Lindelöf

Star-CS Star-CE and Star-CCC

$$
\begin{array}{ccccc}
 & & star - countable & & \\
 & & \downarrow & & \\
 & & star - CS & & \\
 & \swarrow & & \searrow & \\
star - Lindelöf & & & & \\
\downarrow & & & & star - CCC \\
star - CE & & & & \\
 & \searrow & & \swarrow & \\
 & & feebly\ \ Lindelöf & &
\end{array}
$$

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Star-CS Star-CE and Star-CCC

Proposition

There is a Hausdorff star-CS space which is not star-countable.

Star-CS Star-CE and Star-CCC

Proposition

There is a Hausdorff star-CS space which is not star-countable.

A measurable set $A \subset \mathbb{R}$ has density d at $x \in \mathbb{R}$ if

$$\lim_{h \to 0} \frac{m\left(A \cap [x-h, x+h]\right)}{2h}$$

exists and equals d. Denote $\phi(A) = \{x \in \mathbb{R} : d(x, A) = 1\}$ and let τ_d be the family of all measurable sets A such that $A \subset \phi(A)$. τ_d is a non-normal Tychonoff topology on $\mathbb{R}$ which is stronger than the usual topology.

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Proposition

$[CH]$ The space $(\mathbb{R}, \tau_d)$ is a Tychonoff star-CS space wich is not star-countable.

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- The star-CS, star-CE and star-CCC properties are not necessarily inherited by closed sets or closed G_δ-sets or zero sets.

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- The star-CS, star-CE and star-CCC properties are not necessarily inherited by open sets or dense subspaces nor even by an open dense subspace.

Proposition

The star-CS, star-CE and star-CCC properties are inherited over open F_σ-sets.

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Proposition

The star-CS, star-CE and star-CCC properties are inherited over open F_σ-sets.

Corollary

The star-CS, star-CE and star-CCC properties are inherited over clopen sets.

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The star-CS, star-CE and star-CCC properties are inherited over cozero sets.

Example

Let $\mathcal{A}$ be a MAD family in ω, with $|\mathcal{A}| = 2^{\omega}$, and let αD be the one-point compactification of the discrete space D of size 2^{ω}. The space $X_1 = \mathcal{A} \cup (\omega \times \alpha D)$, where $\omega \times \alpha D$ is an open subspace of X_1 and an open neighborhood for $A \in \mathcal{A}$ takes the form

$$O_{F,U}(A) = \{A\} \cup ((A \setminus F) \times U)$$

where $F \subset A$ is finite and $U \subset \alpha D$ is open, is a non star-CCC Tychonoff space.

Proposition

There is a Tychonoff star-countable space with a regular closed set non star-CCC.

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Proposition

There is a Tychonoff star-countable space with a regular closed set non star-CCC.

Proof.

Considering the non star-CCC space $X_1 = \mathcal{A} \cup (\omega \times \alpha D)$ and the psi-space $X_2 = \Psi(\mathcal{A})$ construct a quotient space X resulting from identify the subspace $\mathcal{A}$ of X_1 with the subspace $\mathcal{A}$ of X_2. The space X is a star-countable space and has an homeomorphic copy of X_1 as a regular closed subset. □

Proposition

There is a Tychonoff star-countable space with a regular closed set non star-CE.

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Proposition

There is a Tychonoff star-countable space with a regular closed set non star-CE.

Proposition

There is a Lindelöf space with a regular open set which is neither a star-CE space nor star-CCC space.

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Proposition

If X is star-CE and K is a compact space then $X \times K$ is star-CE.

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Example

The space $X = \Psi(\mathcal{A}) \times \alpha D$ is not star-CCC despite of being the product of a separable space and a compact space.

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Example

The space $X = \Psi(\mathcal{A}) \times \alpha D$ is not star-CCC despite of being the product of a separable space and a compact space.

Proposition

If X is star-CCC (star-CS) and K is a compact separable space then $X \times K$ is a star-CCC (star-CS) space.

Proposition

If $e(X) > \omega$ and $c(Y) > \omega$ then $X \times Y$ is not a star-CCC space.

Proposition

If $e(X) > \omega$ and $c(Y) > \omega$ then $X \times Y$ is not a star-CCC space.

Problem

Is the product of a star-CCC (star-CS) space and a compact CCC space, star-CCC (star-CS)?

Proposition

There exist a countably compact space X and a Lindelöf space Y such that $X \times Y$ is neither star-CE nor star-CCC.

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Proposition

There exist a countably compact space X and a Lindelöf space Y such that $X \times Y$ is neither star-CE nor star-CCC.

Proposition

There exist two countably compact spaces X and Y such that $X \times Y$ is neither star-CE nor star-CCC.

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Proposition

There exist two Lindelöf spaces X and Y such that $X \times Y$ is neither star-CE nor star-CCC.

A subset $A \subset \mathbb{R}$ is *totally imperfect* if A does not contain a copy of the Cantor set.

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A subset $A \subset \mathbb{R}$ is *totally imperfect* if A does not contain a copy of the Cantor set.

Theorem (Bernstein)

There exists $A \subset \mathbb{R}$ such that $|A| = |\mathbb{R} \setminus A| = 2^\omega$ and either A or $\mathbb{R} \setminus A$ are totally imperfect sets.

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Example

Define a finner topology than the Euclidian on $Y = A \cup (\mathbb{R} \setminus A)$ by isolating each point in $\mathbb{R} \setminus A$. Y endowed with this topology is a Lindelöf space. Take $X = A$ with the subspace topology and call $M = X \times Y$.

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Proposition

The space M has the following properties:

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1. M is the product of a second countable space with a Lindelöf space.

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2. M is not star-CCC.

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3. M is star-Lindelöf.

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3. M is star-Lindelöf.

Problem

Is the product of a star-CE and second countable space, star-CE?

Star-CS Star-CE and Star-CCC

Relations with other covering properties

$$\begin{array}{ccccc}
\text{Lindelöf} & & & & \text{separable} \\
 & \searrow & & \swarrow & \\
 & & \text{star}-\text{countable} & & \\
 & & \downarrow & & \\
 & & \text{star}-CS & & \\
 & \swarrow & & \searrow & \\
\text{star}-\text{Lindelöf} & & & & \\
\downarrow & & & & \text{star}-CCC \\
\text{star}-CE & & & & \\
 & \searrow & & \swarrow & \\
 & & \text{feebly Lindelöf} & &
\end{array}$$

Definition

A space X is paralindelöf (σ-paralindelöf) if every open cover of X has a locally countable (σ-locally countable) open refinement.

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Proposition (Blair)

Every paralindelöf feebly Lindelöf space is Lindelöf.

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Problem

Does every σ-paralindelöf feebly Lindelöf space is Lindelöf?

Proposition (Hiremath)

If X is σ-paralindelöf then X is Lindelöf if and only if X is star-Lindelöf.

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Proposition

If X is σ-paralindelöf then X is Lindelöf if and only if X is star-CCC.

Proposition

If X is σ-paralindelöf and contains a dense subspace with countable extent then X is Lindelöf.

Proposition

If X is σ-paralindelöf and contains a dense subspace with countable extent then X is Lindelöf.

Corollary

If X is σ-paralindelöf then X is Lindelöf if and only if X is star-CE.

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Proposition

If X is σ-paralindelöf and contains a dense subspace with countable extent then X is Lindelöf.

Corollary

If X is σ-paralindelöf then X is Lindelöf if and only if X is star-CE.

Proof.

If $\mathcal{U}$ is an open cover and $M \subset X$ is a star kernel of $\mathcal{U}$ with $e(M) \leq \omega$, then $cl_X M$ is Lindelöf and is a star kernel of $\mathcal{U}$. Thus X is Lindelöf. □

Corollary

Let X be a σ-paralindelöf space. The following are equivalent:

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Corollary

Let X be a σ-paralindelöf space. The following are equivalent:

i) X is Lindelöf

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ii) X is star-CS

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i) X is Lindelöf
ii) X is star-CS
iii) X is star-CE

Corollary

Let X be a σ-paralindelöf space. The following are equivalent:

i) X is Lindelöf
ii) X is star-CS
iii) X is star-CE
iv) X is star-CCC

Theorem (Alas, Junqueira, van Mill, Tkachuk, Wilson, 2011)

If X is a Moore space, then X is separable if and only if X is star-Lindelöf.

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Corollary

If X is a Moore space, then X is separable if and only if X is star-CE.

Theorem (Alas, Junqueira, van Mill, Tkachuk, Wilson, 2011)

If X is a Moore space, then X is separable if and only if X is star-Lindelöf.

Corollary

If X is a Moore space, then X is separable if and only if X is star-CE.

Proposition

If X is a semistratifiable space, then X is star-countable if and only if X is star-CE.

The hyperspace $\mathcal{F}[\mathbb{R}]$ with the Pixley-Roy topology is a Moore CCC space which is not star-CE.

The hyperspace $\mathcal{F}[\mathbb{R}]$ with the Pixley-Roy topology is a Moore CCC space which is not star-CE.
When is the hyperspace $\mathcal{F}[X]$ a star-CE space?

For any space X, $\mathcal{F}[X]$ has the following properties:

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For any space X, $\mathcal{F}[X]$ has the following properties:
i) $\mathcal{F}[X]$ is a hereditarily metacompact space.
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iii) $\mathcal{F}[X]$ is a semistratifiable space if and only if $\psi(X) \leq \omega$.
iv) $d(\mathcal{F}[X]) = L(\mathcal{F}[X]) = e(\mathcal{F}[X]) = |X|$.

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Proposition

$\mathcal{F}[X]$ is star-countable if and only if $|X| \leq \omega$.

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Proposition

$\mathcal{F}[X]$ is star-countable if and only if $|X| \leq \omega$.

Proposition

If $\mathcal{F}[X]$ is star-CE then X is hereditarily Lindelöf.

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Proposition

$\mathcal{F}[X]$ is star-countable if and only if $|X| \leq \omega$.

Proposition

If $\mathcal{F}[X]$ is star-CE then X is hereditarily Lindelöf.

Corollary

$\mathcal{F}[X]$ is star-CE if and only if $|X| \leq \omega$.

Weakly star-P properties

Definition

A space X is weakly Lindelöf if for every open cover $\mathcal{U}$ of X there is a countable subset $\mathcal{U}_0 \subset \mathcal{U}$ such that $\bigcup \mathcal{U}_0$ is dense in X.

Weakly star-P properties

Definition

A space X is weakly Lindelöf if for every open cover $\mathcal{U}$ of X there is a countable subset $\mathcal{U}_0 \subset \mathcal{U}$ such that $\bigcup \mathcal{U}_0$ is dense in X.

Proposition (Sakai)

If $\mathcal{F}\left[X\right]$ is weakly Lindelöf, then X is hereditarily Lindelöf. In addition, if $t\left(X\right) \leq \omega$ then X is hereditarily separable.

Weakly star-P properties

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Definition

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Weakly star-P properties

weakly Lindelöf $\Rightarrow$ weakly star-countable

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Proposition

If X is weakly star-countable and metalindelöf then X is weakly Lindelöf.

Weakly star-P properties

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If $\mathcal{F}[X]$ is weakly star-CE, then $\mathcal{F}[X]$ is weakly star-countable.

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If $\mathcal{F}[X]$ is weakly star-CE, then $\mathcal{F}[X]$ is weakly star-countable.

Corollary

If $\mathcal{F}[X]$ is weakly star-CE, then X is hereditarily Lindelöf. In addition, if $t(X) \leq \omega$ then X is hereditarily separable.

Thank you