On the shadowing property and odometers

(joint work with J. Li)

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Basic setting

1. X always compact metric space
2. $\#X > 1$ - nondegenerate
3. $T \colon X \to X$, always continuous
4. (X, T) - a dynamical system

Poulsen simplex

1. $M_T(X)$ - set of all T-invariant probablility measures
2. $M_T(X)$ is compact (and metrizable) in weak*-topology
3. ergodic measures are extreme points of $M_T(X)$
4. when ergodic measures are dense in $M_T(X)$ then $M_T(X)$ is singleton or infinite (so-called Poulsen simplex; 1961).
5. Poulsen simplex is unique up to affine homeomorphism (Lindenstrauss, Olsen, Sternfel; 1978).
6. Any $M_T(X)$ is affine homeomorphic to $M_S(X)$ for some Toeplitz minimal subshift (Y, S) in Σ_2 (Downarowicz, 1991).

Specification property

Definition

We say that (X, T) has the specification property if for any $\varepsilon > 0$ there is a constant $N = N(\varepsilon) \in \mathbb{N}$ such that for any $a_1 \leq b_1 < \ldots < a_n \leq b_n$ with $b_{i+1} - a_i \geq N$ and any $x_1, \ldots, x_n \in X$ there is y such that $d(T^i(y), T^i(x_j)) < \varepsilon$ provided that $a_j \leq i \leq b_j$.

- If additionally, y can be chosen in such a way that $T^{b_n - a_0 + N}(y) = y$ then (X, T) has the periodic specification property.

1. If (X, T) has specification property then every invariant measure can be arbitrarily close approximated by an ergodic measure [Sigmund, 1970s].
2. For periodic specification: measures on periodic orbits.
3. Mixing map on topological graph has periodic specification property [Blokh].

Invariant measures and topological entropy

1. By Variational Principle:

$$
\begin{aligned}
h_{\text{top}}(T) &= \sup_{\mu \in M_T(X)} h_\mu(T) \\
&= \sup_{\{\mu \text{ - ergodic}\}} h_\mu(T)
\end{aligned}
$$

2. If there exists $\mu \in M_T(X)$ such that

$$h_{\text{top}}(T) = h_\mu(T)$$

then μ is so-called measure of maximal entropy (m.m.e. for short).

3. Suppose m.m.e. exists. By ergodic decomposition theorem of entropy

$$h_\mu(T) = \int_{\nu \text{ - ergodic}} h_\nu(T) d\tau$$

if $h_{\text{top}}(T) < \infty$ then there is ergodic measure of max. entropy.

Specification property and ergodic measures

1. When interval map is piecewise monotone (finitely many pieces) or C^∞ then there is measure of maximal entropy [Hofbauer, Buzzi]
2. There are mixing C^r interval maps without measure of maximal entropy.
3. Measure $\mu \in M_T(X)$ is entropy approachable by ergodic measures, if for every $h^* < h_\mu(T)$ and every neighborhood U of μ there is an ergodic measure $\nu \in U$ with $h_\nu(T) > h^*$.
4. (X, T) is entropy dense if every invariant measure is entropy approachable.
5. If (X, T) has the specification property then it is entropy-dense [Eizenberg, Kifer, Weiss, 1994].
6. If $\mu \mapsto h_\mu(T)$ is upper semicontinuous then a measure of maximal entropy exists.

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Upper semicontinuous entropy function

1. There exists minimal (X, T) such that
   - $M_T(X)$ is Poulsen simplex
   - there is unique measure $\mu \in M_T(X)$ with $h_\mu(T) > 0$.
   - [Gelfert, Kwietniak, 2015]
2. note that $\mu \mapsto h_\mu(T)$ is upper semicontinuous in that example

Shadowing property

1. a finite sequence $x_1, \ldots, x_n$ is δ-pseudo orbit if $d(T(x_i), x_{i+1}) < \delta$ for $i = 1, \ldots, n-1$
2. a point z ε-traces δ-pseudo orbit if $d(T^i(z), x_i) < \varepsilon$.
3. (X, T) has shadowing property if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit can be ε-traced.
4. shadowing property + topological mixing $\Longrightarrow$ specification property
    - but not necessarily periodic specification property (maybe no periodic points in dynamics at all)
    - not necessarily expansive

Where to find shadowing?

1. Classical case: hyperbolic dynamics.
   1. shadowing property coexists with some form of expansivity/expanding.
   2. so entropy function is upper semicontinuous.
2. Generic case: in many cases (e.g. manifolds with triangulation) shadowing property is generic.
   1. Expansivity is not a generic property,
   2. neither is transitivity.
3. Dimension one: many examples; in the family of tent maps for almost all parameters.
   1. Main forms of expansivity are not possible.
   2. These examples are transitive on the core.

Results of Pfister and Sullivan - a closer look

Definition

We say that a dynamical system (X, T) has the approximate product structure if for any $\varepsilon > 0$, $\delta_1 > 0$ and $\delta_2 > 0$ there exists an integer $N > 0$ such that for any $n \geq N$ and $\{x_i\}_{i=1}^{\infty} \subset X$ there are $\{h_i\}_{i=1}^{\infty} \subset \mathbb{N}$ and $y \in X$ satisfying $h_1 = 0$, $n \leq h_{i+1} - h_i \leq n(1 + \delta_2)$ and

$$\left|\left\{0 \leq j < n \,:\, \rho\left(T^{h_i+j}(y), T^j(x_i)\right) > \varepsilon\right\}\right| \leq \delta_1 n \text{ for all } i \in \mathbb{N}.$$

Theorem (Pfister & Sullivan)

If (X, T) has approximate product property then ergodic measures are entropy dense.

Proposition (Kwietniak, Łącka, O.)

If transitive (X, T) has shadowing property, then it has approximate product property.

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Recent results with Jian Li (unpublished; in preprint form)

Let μ be any invariant measure for (X, T)

- When (X, T) has approximate product property then:
  1. $\lim_{n\to\infty} \mu_n = \mu$,
  2. $\liminf_{n\to\infty} h_{\mu_n}(T) \geq h_\mu(T)$,
  3. for some ergodic μ_n (supp μ_n is ... ?).
- If additionally $\mu \mapsto h_\mu(T)$ is upper semicontinuous then:
  1. $\limsup_{n\to\infty} h_{\mu_n}(T) \leq h_\mu(T)$, so $\lim_{n\to\infty} h_{\mu_n}(T) = h_\mu(T)$.

Theorem (Li, O.)

Suppose that (X, T) has shadowing property and is transitive. In this case:

1. *invariant measures whose supports are odometers (this includes periodic orbits) are dense in $M_T(X)$.*
2. *there is a sequence of ergodic measures μ_n such that:*
   1. *support of each μ_n is almost 1-1 extension of an odometer,*
   2. $\lim_{n\to\infty} \mu_n = \mu$,
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A final remark

Example (Li, O.)

There exists subshift (X, T) in Hilbert cube $[0,1]^{\mathbb{N}}$ which is transitive and has shadowing property but does not have measure of maximal entropy.

In particular $\mu \mapsto h_\mu(T)$ is not upper semicontinuous.