Group compactifications and Ramsey-type phenomena

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Toposym 2016

Outline

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- The Kechris-Pestov-Todorcevic correspondence.
- Making the KPT correspondence broader: two examples.
- Making the KPT correspondence broader: the general framework

Part I

The KPT correspondence

Extremely amenable groups

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Theorem (Herrer-Christensen, 75)

There is a Polish Abelian extremely amenable group.

Theorem (Veech, 77)

Let G be non-trivial and locally compact.
Then G is not extremely amenable.

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$$d(f,g) = \int_0^1 d(f(x), g(x)) d\mu.$$

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Remark

Examples 3, 4, and 5 by Pestov use some Ramsey theoretic results.

The KPT correspondence

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Theorem (Kechris - Pestov - Todorcevic, 05)

There is a link between extreme amenability and Ramsey theory when G is a closed subgroup of S_∞.

Definition

S_∞: the group of permutations of $\mathbb{N}$.
Basic open sets: $f \in S_\infty$, $F \subset \mathbb{N}$ finite.

$$U_{f,F} = \{g \in S_\infty : g \upharpoonright F = f \upharpoonright F\}.$$

This topology is Polish.

Fact

The closed subgroups of S_∞ are exactly the automorphism groups of countable ultrahomogeneous first order structures...

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Examples

$\mathbb{N}$, $(\mathbb{Q}, <)$, *the random graph, the dense local order* $S(2)$, *the countably-dimensional vector space over a given finite field, the countable atomless Boolean algebra,...*

Every countable ultrahomogeneous structure $\mathbb{F}$ is attached to:

- $Age(\mathbb{F})$ the set of finite substructures of $\mathbb{F}$.
- $Aut(\mathbb{F}) \leq S_\infty$.

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The KPT correspondence expresses combinatorially, at the level of $Age(\mathbb{F})$, when $Aut(\mathbb{F})$ is extremely amenable.

Definition

A class $\mathcal{K}$ of finite structures has the Ramsey property *when for any $A, B \in \mathcal{K}$, $k \in \mathbb{N}$ there is $C \in \mathcal{K}$ so that:*

Whenever embeddings of A in C are colored with k colors,
there is $\tilde{B} \cong B$ where all embeddings of A have same color.

When $\mathcal{K} = Age(\mathbb{F})$:

Whenever embeddings of A in $\mathbb{F}$ are colored with finitely many colors,
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Examples

- *First example: $Age(\mathbb{Q}, <)$ (Ramsey, 30)*
- *Boolean algebras (Graham-Rothschild, 71)*
- *Vector spaces over finite fields (Graham-Leeb-Rothschild, 72)*
- *Relational structures (Nešetřil-Rödl, 77 ; Abramson-Harrington, 78)*
- *Relational struct. with forbidden configurations (Nešetřil-Rödl, 77-83)*
- *Posets (Nešetřil-Rödl, ∼83; published by Paoli-Trotter-Walker, 85))*
- ...

Theorem (Kechris-Pestov-Todorcevic, 05)

Let $\mathbb{F}$ be a countable ultrahomogeneous structure. TFAE:

i) *$Aut(\mathbb{F})$ is extremely amenable.*

ii) *$Age(\mathbb{F})$ has the Ramsey property.*

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- Aforementioned Ramsey-type results led to numerous extremely amenable groups of the form $Aut(\mathbb{F})$ (e.g.: $Aut(\mathbb{Q}, <)$), but not only (e.g. $Homeo_{+}([0, 1])$, $iso(\mathbb{U})$).

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- New motivation to prove Ramsey-type results, see work by: Bartosova-Kwiatkowska, Bartosova-Lopez-Abad-Mbombo, Bodirsky, Dorais et al., Foniok, Foniok-Böttcher, Jasiński, Jasiński-Laflamme-NVT-Woodrow, Kechris-Sokić, Kechris-Sokić-Todorcevic, Nešetřil, Nešetřil-Hubička, NVT, Sokić, Solecki, Solecki-Zhao,...

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- Explicit description of various dynamical objects, among which universal minimal flows.

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- Good news: There is such a correspondence.
  Goal of this talk: Convince that Ramsey-type properties naturally appear when expressing combinatorially the existence of fixed points in certain compactifications.
- Bad news: Very unclear whether this correspondence will be as useful as the original KPT in practice.

Part II

Making the KPT correspondence broader: two examples

Some natural classes of flows to start with

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Definition

Let $G \curvearrowright X$ be a G-flow. An ordered pair $(x, y) \in X^2$ is:

- *proximal when $g \cdot x$ and $g \cdot y$ can be made arbitrarily close.*
- *distal when it is not proximal.*

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- *distal when it is not proximal.*

Definition

A G-flow $G \curvearrowright X$ is:

- *proximal when every $(x, y) \in X^2$ is proximal.*
- *distal when every $(x, y) \in X^2$ with $x \neq y$ is distal.*
- *equicontinuous when*

$$\forall U \in Unif(X) \ \ \exists V \in Unif(X) \ \ \forall x, y \in X$$

$$(x, y) \in V \Rightarrow \forall g \in G \ \ (g \cdot x, g \cdot y) \in U$$

Fixed-points properties

Definition

Let G be a topological group. It is:

- *strongly amenable when every proximal G-flow has a fixed point.*
- *minimally almost periodic when every equicontinuous G-flow has a fixed point ("equicontinuous "may be replaced by "distal").*

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Remark

Recall that a topological group G is amenable when every G-flow has an invariant Borel probability measure.

- *Amenability is also a fixed point property: G is amenable iff every G-flow $G \curvearrowright X$ has a fixed point, provided $G \curvearrowright \mathrm{Prob}(X)$ is proximal.*
- *Thus, every strongly amenable group G is amenable.*
- *KPT correspondence for amenability already considered by Tsankov and by Moore (∼10). It is of slightly different flavor than what follows, probably more useful in practice (even if not used so far).*

Proximal colorings

Definition

Let $\mathbb{F}$ be a countable ultrahomogeneous structure and $A \in Age(\mathbb{F})$.
A finite coloring χ of the embeddings of A in $\mathbb{F}$ is proximal when:
For every $(g_m)_{m\in\mathbb{N}}, (h_m)_{m\in\mathbb{N}} \in Aut(\mathbb{F})$ that satisfy

$$(\chi(g_m \cdot a))_m, (\chi(h_m \cdot a))_m \text{ converge for every } a,$$

There is $B \in Age(\mathbb{F})$ s.t. every $\tilde{B} \cong B$ contains some $\tilde{a}$ s.t. :

$$\lim_m \chi(g_m \cdot \tilde{a}) = \lim_m \chi(h_m \cdot \tilde{a})$$

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Definition

A countable ultrahomogeneous structure $\mathbb{F}$ has the proximal Ramsey property *when: For every $A, B \in Age(\mathbb{F})$,*

Whenever embeddings of A in $\mathbb{F}$ are colored via a proximal finite coloring $\exists \tilde{B} \cong B$ where all embeddings of A have same color.

A (half) KPT correspondence for proximal flows

Theorem (NVT, 15)

Let $\mathbb{F}$ be a countable ultrahomogeneous structure so that $Aut(\mathbb{F})$ is strongly amenable. Then $\mathbb{F}$ has the proximal Ramsey property.

Definable colorings

Definition

Let $\mathcal{K}$ be a class of finite structures.

- *A (joint embedding) pattern $\langle a, z \rangle$ is a pair of embeddings of $A, Z \in \mathcal{K}$ into some common $C \in \mathcal{K}$.*
- *Write $\langle a, z \rangle \cong \langle a', z' \rangle$ when there is an isomorphism $c : C \to C'$ s.t.:*

$$a' = c \circ a, \quad z' = c \circ z$$

- *Fix $A, C, Z \in \mathcal{K}$. A pattern $\langle c, z \rangle$ induces a coloring of the embeddings of A in C:*

$$\chi(a) = \text{isomorphism type of } \langle a, z \rangle.$$

(keeps track of how "a sees z").

- *Also makes sense in case of finitely many $Z^1,\ldots,Z^k$.*
- *Colorings that are obtained that way are definable.*

Definable Ramsey property, stable Ramsey property

Definition

A class of finite structures $\mathcal{K}$ *has the* *definable Ramsey property* *when:*
For every $A, B \in \mathcal{K}$*, every* $Z^1, ..., Z^k \in \mathcal{K}$*, there exists* $C \in \mathcal{K}$ *s. t.*

$$\text{Whenever embeddings of } A \text{ in } C \text{ are colored via some } \langle c, z^1, ..., z^k \rangle,$$
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Whenever embeddings of A in C are colored via some $\langle c, z^1, ..., z^k \rangle$, $\exists \tilde{B} \cong B$ where all embeddings of A have same color.

Definition

$\mathcal{K}$ has the stable Ramsey property when the definable Ramsey property is restricted to those $A, Z^1, ..., Z^k$ with all (A, Z^i) stable...

...where (A, Z) is stable when there is no $(a_m, z_m)_{m \in \mathbb{N}}$ and no pattern $\langle a, z \rangle$ s.t.:

$$\forall m, n \in \mathbb{N} \quad m < n \Leftrightarrow \langle a_m, z_n \rangle \cong \langle a, z \rangle$$

A KPT correspondence for equicontinuous/distal flows

Theorem (NVT, 15)

Let $\mathbb{F}$ be a countable ultrahomogeneous structure.
Assume that that every pair of elements of $Age(\mathbb{F})$ only has finitely many joint embedding patterns (equiv. $Aut(\mathbb{F})$ is Roelcke precompact). TFAE:

i) *$Aut(\mathbb{F})$ is minimally almost periodic.*

ii) *$Age(\mathbb{F})$ has the stable Ramsey property.*

Part III

Making the KPT correspondence broader: the general framework

Main ideas

- Express the existence of fixed points in G-flows in terms of continuous functions.
- Specialize this to Gelfand compactifications.
- When $G = Aut(\mathbb{F})$, discretize to obtain a Ramsey-type property.
- Use this and additional properties to characterize fixed point properties.

Fixed points in G-flows

Proposition

Let G be a topological group, $G \curvearrowright X$ a G-flow, and $x \in X$. TFAE:

i) $\overline{G \cdot x}$ *contains a fixed point.*

ii) *For every $\mathcal{F} \subset C(X)$ finite, $\varepsilon > 0$, $F \subset G$ finite,*

there exists a point in $G \cdot x$ is F-fixed up to $\mathcal{F}, \varepsilon$

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$$\exists g \in G \quad \forall h, h' \in F \quad \forall f \in \mathcal{F} \quad |f(h \cdot (g \cdot x)) - f(h' \cdot (g \cdot x))| < \varepsilon \quad (*)$$

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Proof.

i)⇒ii): Approximate the fixed point by some point in $G \cdot x$.

ii)⇒i): Use compactness to obtain a true fixed point in $\overline{G \cdot x}$. □

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Remark

$(*)$ *in ii) can be rephrased if we write $f_x : g \mapsto f(g \cdot x)$:*

$$\exists g \in G \quad \forall f \in \mathcal{F} \quad f_x \text{ is constant on } Fg \text{ up to } \varepsilon$$

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Remark

- *Every f_x is in $RUC_b(G)$ (C^*-alg of bdd unif conti fns $(G, \mathcal{U}_R) \to \mathbb{C}$).*

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- *If $\mathcal{A}$ is a unital subalgebra of $RUC_b(G)$ that is invariant under*

$$g \cdot f(x) = f(g^{-1} \cdot x)$$

the action $G \curvearrowright G$ by left translations extends continuously to

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- *Furthermore, if $x = e_G$, then $\{f_x : f \in C(G^{\mathcal{A}})\} = \mathcal{A}$.*
- *So the previous proposition applied to the G-flow $G \curvearrowright G^{\mathcal{A}}$ gives:*

Fixed points in G-flows: Gelfand compactifications

Proposition

Let G be a top. gp, $\mathcal{A}$ a unital, left-invariant subalg of $RUC_b(G)$. TFAE:

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- For finite $A \subset \mathbb{F}$, the ptwise stabilizer $Stab(A) \subset Aut(\mathbb{F})$ is clopen.

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iii) $\mathbb{F}$ has the Ramsey property for colorings in $\mathcal{A}$:
For every $A, B \in Age(\mathbb{F})$, $\mathcal{F}$ finite set of colorings of $\binom{\mathbb{F}}{A}$ st $\mathcal{F} \subset \mathcal{A}$.

$$\exists \tilde{B} \cong B \;\; \forall \chi \in \mathcal{F} \;\; \text{all embeddings of } A \text{ have same } \chi\text{-color.}$$

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...and from previous slides, these are equivalent to:

i) $G \curvearrowright G^{\mathcal{A}}$ has a fixed point.

Fixed points in Gelfand compactifications

Theorem (NVT, 16)

Let $G = Aut(\mathbb{F})$, $\mathcal{A}$ unital, left-invariant subalg of $RUC_b(G)$.
If $G \curvearrowright G^{\mathcal{A}}$ has a fixed point, then $\mathbb{F}$ has the Ramsey property for colorings in $\mathcal{A}$.
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Theorem (NVT, 16)

Let $G = Aut(\mathbb{F})$, $\mathcal{A}$ unital, left-invariant subalg of $RUC_b(G)$. TFAE:

i) *$G \curvearrowright G^{\mathcal{A}}$ has a fixed point.*

ii) *$\mathbb{F}$ has the approximate Ramsey property for colorings in $\mathcal{A}$:*
$\forall A, B \in Age(\mathbb{F})$, $\varepsilon > 0$, $\mathcal{F}$ *finite set of colorings of* $\binom{\mathbb{F}}{A}$ *st* $\mathcal{F} \subset (\mathcal{A})_\varepsilon$

$$\exists \tilde{B} \cong B \ \ \forall \chi \in \mathcal{F} \ \ \text{all embeddings of } A \text{ have same } \chi\text{-color up to } 2\varepsilon.$$

How to apply this in concrete situations

Let (P) be a property of G-flows.
Under identified assumptions, (P) admits a universal object $G \curvearrowright X$:

- $G \curvearrowright X$ has (P).
- Every G-flow with (P) is a factor of $G \curvearrowright X$, ie
  If $G \curvearrowright Y$ has (P), there is $\pi : X \twoheadrightarrow Y$ continuous and equivariant.

Every such object is of the form $G \curvearrowright G^{\mathcal{A}}$.

Examples

- *Being a G-flow.*
- *Being proximal.*
- *Being distal.*
- *Being equicontinuous.*

So: To express that every such G-flow has a fixed point, it suffices to find out the relevant $\mathcal{A}$.

Looking for $\mathcal{A}$

Proposition

Let G be a top gp. The (algebraic) action on $\mathbb{C}^G$ defined by

$$g \cdot f(x) = f(xg)$$

is continuous on every $\overline{G \cdot f}$ for $f \in RUC_b(G)$.
The corresponding G-flow is denoted $G \curvearrowright X_f$.

Proposition (de Vries)

Let (P) be a "good" property of G-flows, attached to $\mathcal{A} \subset RUC_b(G)$.
TFAE for $f \in RUC_b(G)$:

i) $f \in \mathcal{A}$.

ii) $G \curvearrowright X_f$ has (P).

Application: original KPT correspondence

Let $\mathbb{F}$ be a countable ultrahomogeneous structure, $G = Aut(\mathbb{F})$.

- (P): Being a G-flow. This is "good".
- Fixed point property: Extreme amenability.
- $\mathcal{A} = RUC_b(G)$
- Colorings are dense in $\mathcal{A}$.
- So by Theorem, TFAE:

  i) $Aut(\mathbb{F})$ is extremely amenable.
  ii) $Age(\mathbb{F})$ has the Ramsey property for all colorings.

Application: Proximal KPT correspondence

Let $\mathbb{F}$ be a countable ultrahomogeneous structure, $G = Aut(\mathbb{F})$.

- (P): Being a proximal G-flow. This is "good".
- Fixed point property: strong amenability.
- $\mathcal{A} = Prox(G)$. $f \in Prox(G)$ when:

$$\forall (h_n)_n, (h'_n)_n \subset G \quad (h_n \cdot f)_n, (h'_n \cdot f)_n \text{ converge pointwise}$$
$$\Rightarrow \forall \varepsilon > 0 \quad \{g \in G : |\lim_n f(gh_n) - \lim_n f(gh'_n)| < \varepsilon\} \text{ is syndetic}$$

- A finite coloring χ of the embeddings of A in $\mathbb{F}$ is in $Prox(G)$ when: for every $(h_n)_{n\in\mathbb{N}}$, $(h'_n)_{n\in\mathbb{N}} \in Aut(\mathbb{F})$ that satisfy

$$(\chi(h_n \cdot a))_n,\ (\chi(h'_n \cdot a))_n \text{ converge for every } a,$$

there is $B \in Age(\mathbb{F})$ s.t. every $\tilde{B} \cong B$ contains some $\tilde{a}$ s.t. :

$$\lim_n \chi(h_n \cdot \tilde{a}) = \lim_n \chi(h'_n \cdot \tilde{a})$$

- Not clear that colorings are dense in $\mathcal{A}$.
- So if $Aut(\mathbb{F})$ is strongly amenable, then $Age(\mathbb{F})$ has the Ramsey property for proximal colorings.

Application: Distal/equicontinuous KPT correspondence

Let $\mathbb{F}$ be a countable ultrahomogeneous structure, st $G = Aut(\mathbb{F})$ is oligomorphic.

- (P): Being a distal G-flow. This is "good".
- Fixed point property: minimal almost periodicity.
- $\mathcal{A} = Dist(G)$. $f \in Dist(G)$ when:
  $\forall (h_n)_n, (h'_n)_n \subset G \quad (h_n \cdot f)_n, (h'_n \cdot f)_n$ converge ptwise to distinct elts
  $$\Rightarrow \exists \varepsilon > 0 \quad \forall g \in G \quad |\lim_n f(gh_n) - \lim_n f(gh'_n)| \geq \varepsilon$$
- A finite coloring χ of the embeddings of A in $\mathbb{F}$ is in $Dist(G)$ when: for every $(h_n)_{n\in\mathbb{N}}$, $(h'_n)_{n\in\mathbb{N}} \in Aut(\mathbb{F})$ that satisfy
  $$(\chi(h_n \cdot a))_n,\ (\chi(h'_n \cdot a))_n \text{ converge for every } a,$$
  for every $B \in Age(\mathbb{F})$ there is $\tilde{B} \cong B$ where every $\tilde{a}$ satisfies:
  $$\lim_n \chi(h_n \cdot \tilde{a}) \neq \lim_n \chi(h'_n \cdot \tilde{a})$$
- Not clear that colorings are dense in $\mathcal{A}$.
- So if $Aut(\mathbb{F})$ is minimally almost periodic, then $Age(\mathbb{F})$ has the Ramsey property for distal colorings.

BUT...

- ...It is known that replacing $\mathcal{A}$ by another algebra $WAP(G)$, the corresponding fixed point property stays unchanged.

  Thanks to some recent results of Ben Yaacov-Tsankov:
- A finite coloring χ of the embeddings of A in $\mathbb{F}$ is in $WAP(G)$ when it is stable: $\chi(a) = \langle a, z \rangle$ for some stable (A, Z).
- Colorings are dense in $WAP(G)$.

- So by Theorem, TFAE:
  i) $Aut(\mathbb{F})$ is minimally almost periodic.
  ii) $Age(\mathbb{F})$ has the Ramsey property for stable colorings.

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