On the classification of one dimensional continua that admit expansive homeomorphisms.

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Preliminaries

A *continuum* is a compact connected metric space.

A continuum is *1-dimensional* if for every $\epsilon > 0$ there exists a finite open cover $\mathcal{U}$ of X with mesh less than ϵ such that every $x \in X$ is contained in at most 2 elements of $\mathcal{U}$.

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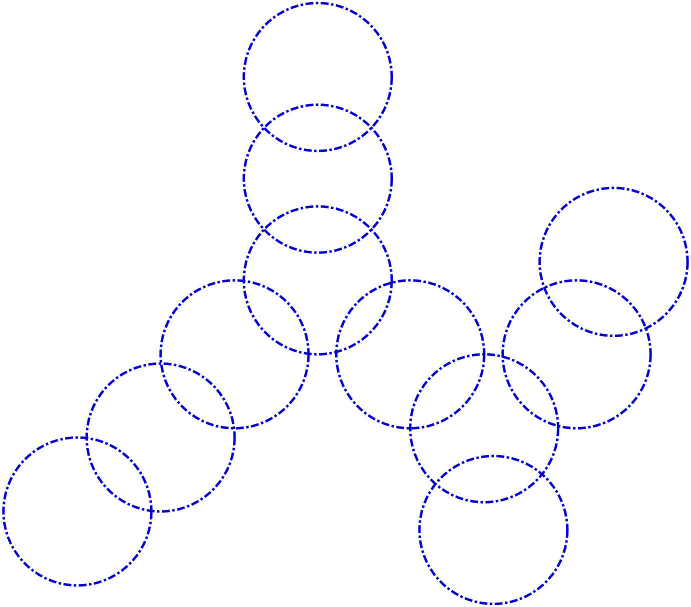

A continuum is

1. *chainable* (also known as *arc-like*)
2. *tree-like*
3. *G-like*
4. *k-cyclic*

if it is the inverse limit of (or if for every $\epsilon > 0$ there exist an open cover whose nerve is a(n))

1. arc(s)
2. tree(s)
3. topological graph(s) homeomorphic to the same graph G
4. topological graph(s) each having at most k distinct simple closed curves

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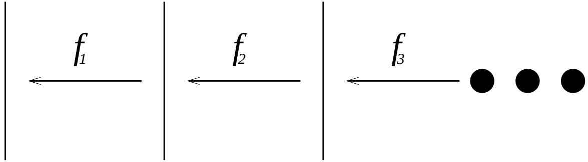

Figure: Arc-like

$$X = \{(x_1, x_2, x_3, ...) \mid f_i(x_{i+1}) = x_i\}.$$

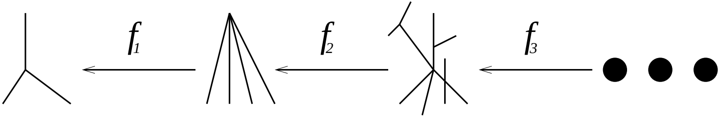

Figure: Tree-like

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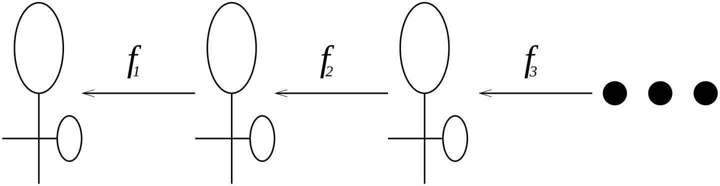

Figure: G-like

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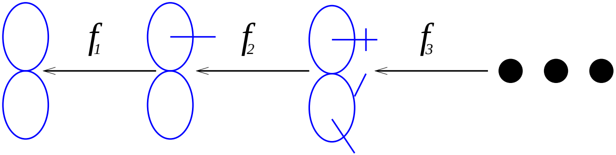

Figure: *k*-cyclic

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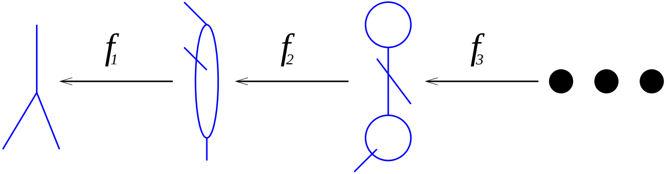

Figure: Not k-cyclic

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Arc$-$like	$\subseteq$	Tree$-$Like
$\rotatebox{90}{$\subseteq$}$		$\rotatebox{90}{$\subseteq$}$
G $-$like	$\subseteq$	k $-$cyclic
		$\rotatebox{90}{$\subseteq$}$
		All 1$-$dimensional continua

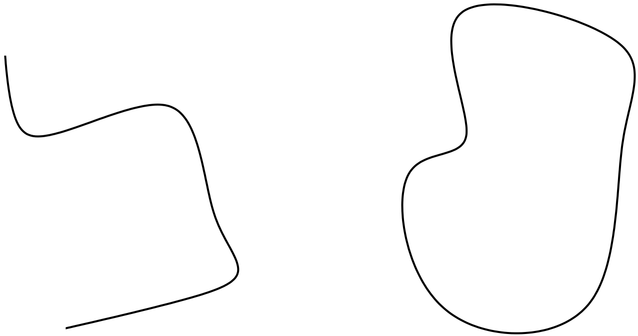


Figure: Arc-like and G-like (circle-like).

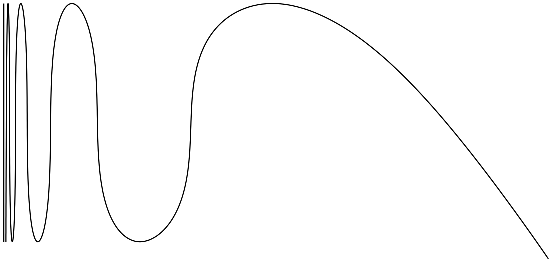


Figure: arc-like

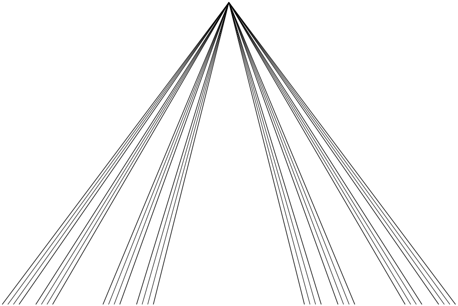


Figure: tree-like

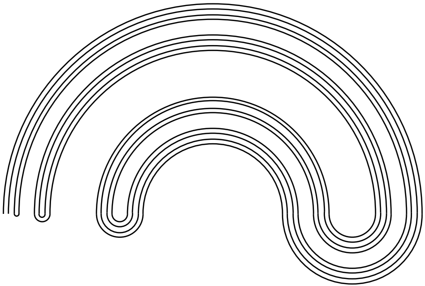


Figure: arc-like

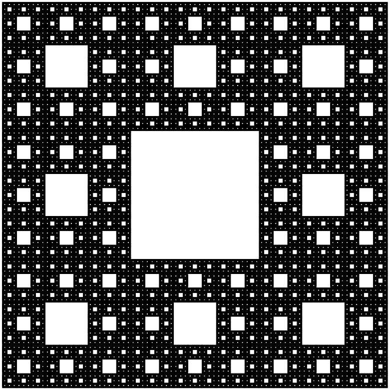


Figure: not k-cyclic

A continuum X is *decomposable* if it is the union of 2 of its **proper** subcontinua.

A continuum is *indecomposable* if it is not decomposable.

Equivalently, X is indecomposable if every **proper** subcontinuum is nowhere dense.

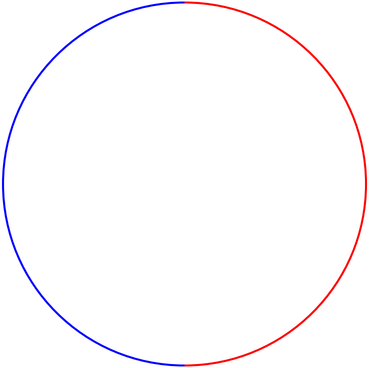
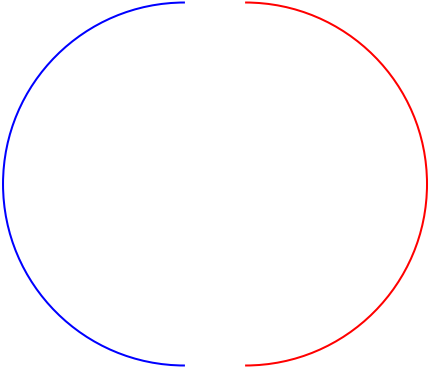


Figure: Decomposable



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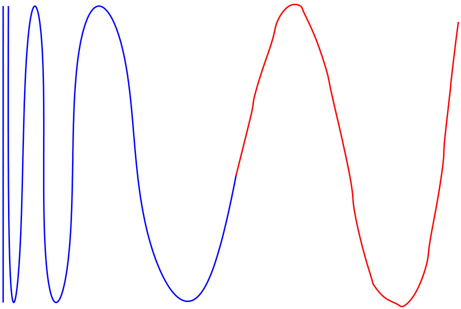

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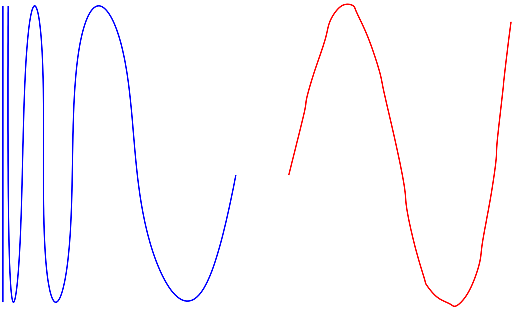

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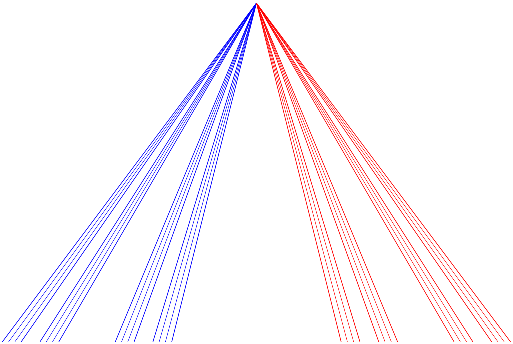

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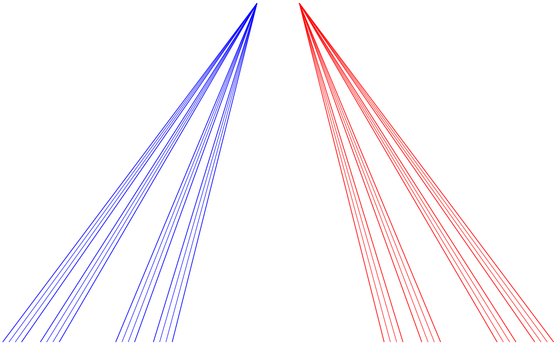

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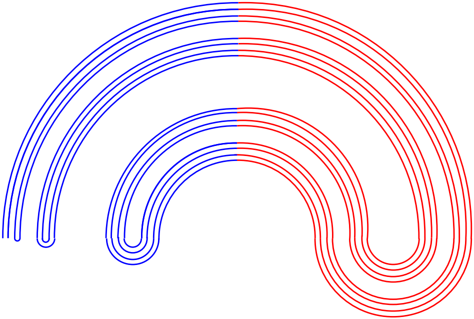
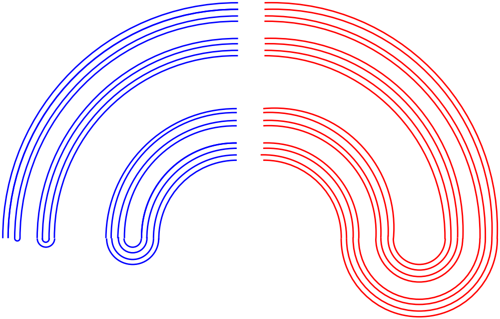


Figure: Indecomposable



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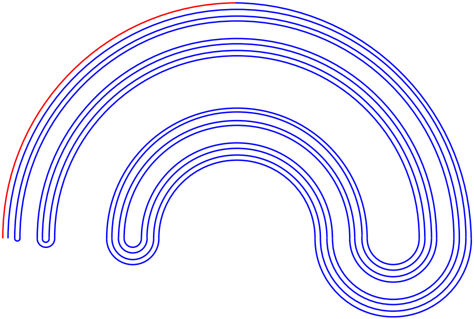

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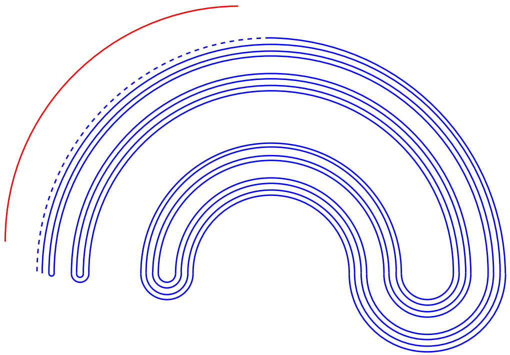

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Expansive Homemorphisms

A homeomorphism $h : X \to X$ is called *expansive* provided that there exists a constant $c > 0$ such that for every $x, y \in X$ there exists an integer n such that $\mathrm{d}(h^n(x), h^n(y)) > c$.

Here, c is called the expansive constant.

Expansive homeomorphisms exhibit sensitive dependence on initial conditions in the strongest sense in that no matter how close any two points are, either their images or pre-images will at some point be at least a certain distance apart.

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Example.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = 2x$ and let the expansive constant be 1.

Suppose that x and y are distinct real numbers. Then there exists an integer n such that

$$2^n|x - y| = |2^n x - 2^n y| = |f^n(x) - f^n(y)| > 1.$$

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Example: *The shift homeomorphism of the dyadic solenoid, Σ_2, is expansive.*

Define $f : S \longrightarrow S$ by $f(x) = 2x \mod 1$. Let $\Sigma_2 = \varprojlim(S, f)$
Define the *shift homeomorphism* $\widehat{f} : \Sigma_2 \longrightarrow \Sigma_2$ by

$$\widehat{f}(\mathbf{x}) = \widehat{f}(\langle x_1, x_2, x_3 \ldots\rangle) = \langle f(x_1), f(x_2), f(x_3), \ldots\rangle = \langle f(x_1), x_1, x_2, \ldots\rangle.$$

Also, notice that

$$\widehat{f}^{-1}(\langle x_1, x_2, x_3 \ldots\rangle) = \langle x_2, x_3, x_4, \ldots\rangle.$$

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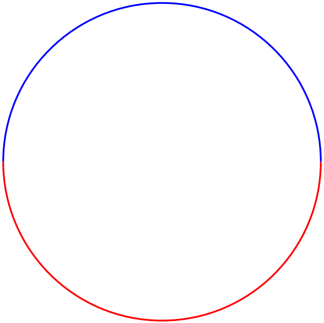
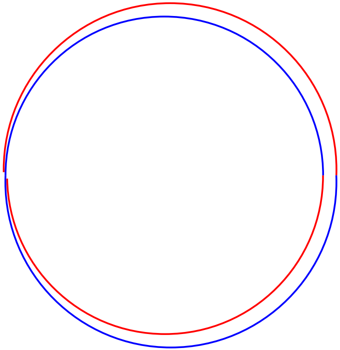


Figure: Doubling map $f(x) = 2x \mod 1$.



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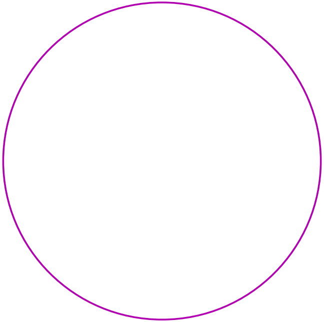

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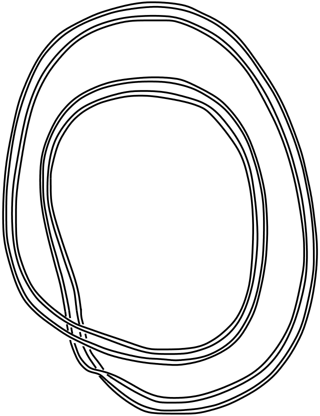

Figure: Inverse limit of $f(z)$ is the solenoid Σ_2.

Theorem

The shift homeomorphism on Σ_2 is expansive.

Proof.

Let the expansive constant be $\frac{1}{4}$.Notice that if $x, y \in S$ and $\mathrm{d}_S(x, y) < \frac{1}{4}$ then $\mathrm{d}_S(f(x), f(y)) = 2\mathrm{d}_S(x, y)$.Let $\mathbf{x}, \mathbf{y}$ be distinct elements in Σ_2. Then there exists i such that $x_i \neq y_i$.Furthermore, there exists a nonnegative natural number n such that $\frac{1}{4} < 2^n \mathrm{d}_S(x_i, y_i) \leq 1/2$. Hence,

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Every expansive homeomorphism is continuum-wise expansive, but the converse is not true.

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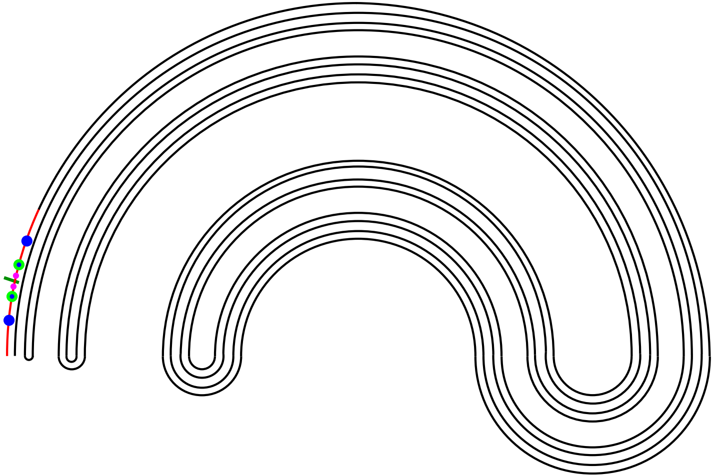
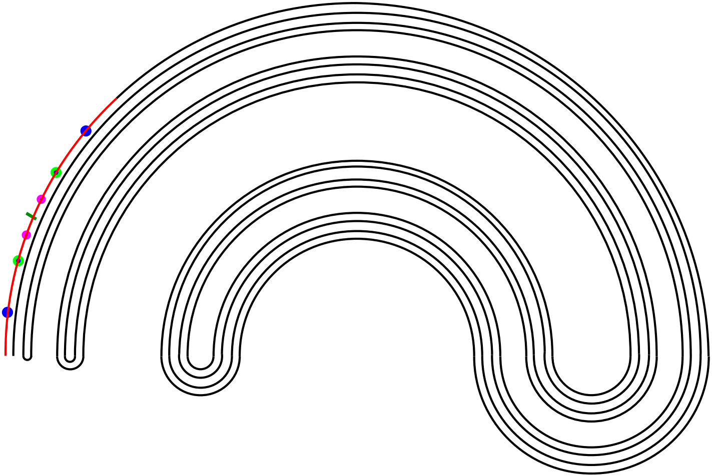
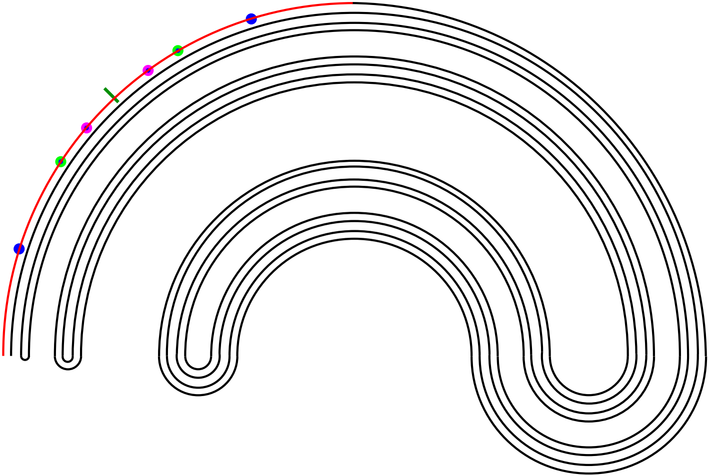
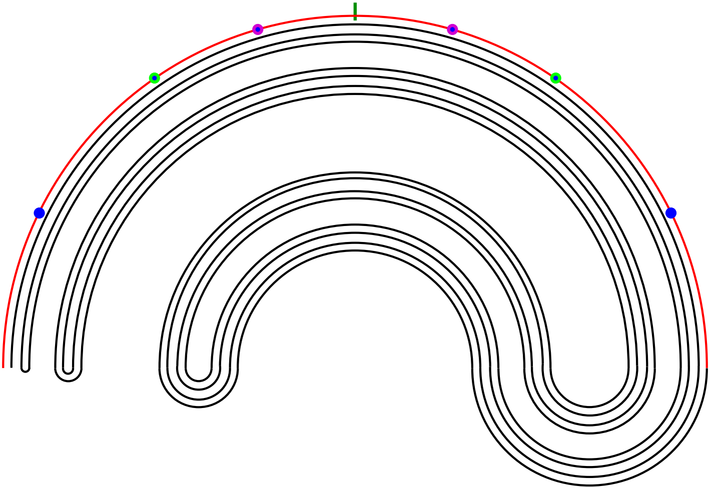
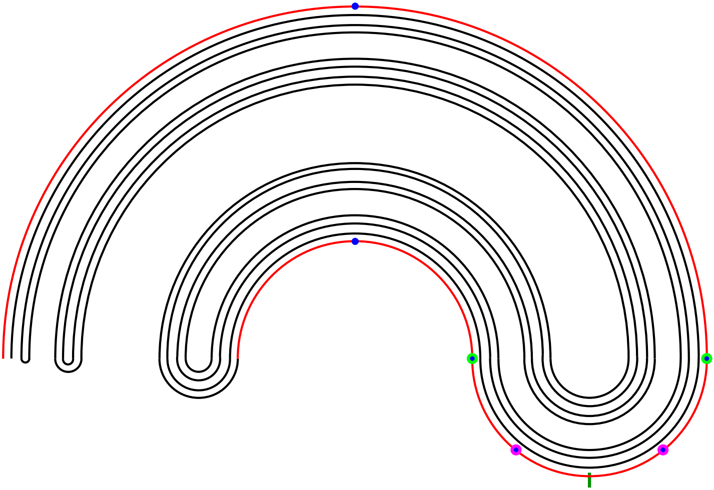
A homeomorphism is *positively continuum-wise fully expansive* if for every pair $\epsilon, \delta > 0$ there is a $N(\epsilon, \delta) > 0$ such that if Y is a subcontinuum of X with $\mathrm{diam}(Y) \geq \delta$, then $\mathrm{d}_H(h^n(Y), X) < \epsilon$ for all $n \geq N(\epsilon, \delta)$.

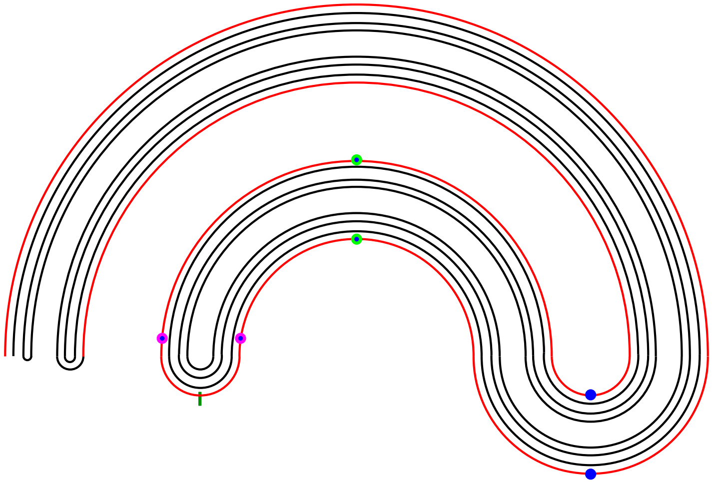


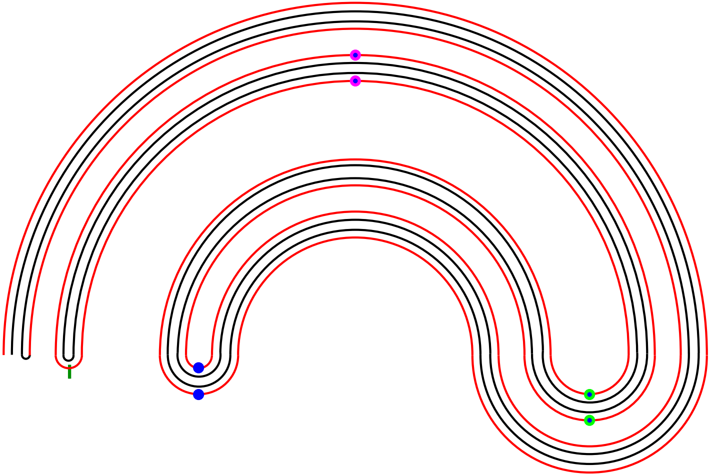


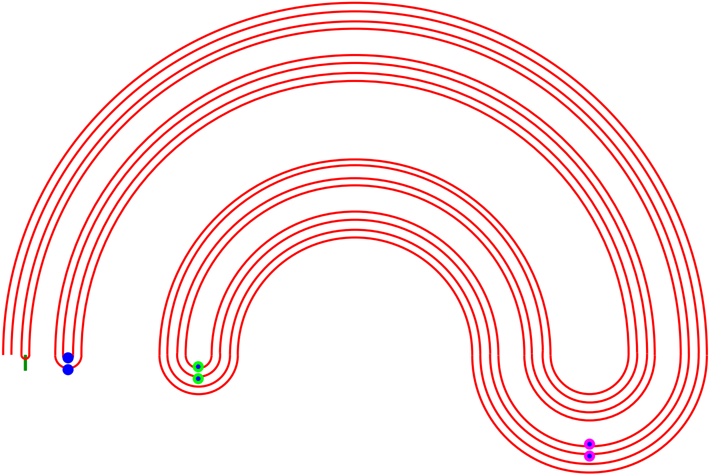












Question

What properties must a continuum have in order for a continuum to admit and expansive (continuum-wise expansive) homeomorphism?

Well, in order for a continuum to admit an expansive (continuum-wise) homeomorphism, all of the proper subcontinuum must be continuous stretched and there must be room for this to happen. This is how indecomposable continua are created.

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Suppose that X is a one-dimensional continuum that admits an expansive homeomorphism, must X be indecomposable?

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Suppose that X is a one-dimensional continuum that admits an expansive homeomorphism, must X be indecomposable?

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Define $f : S \longrightarrow S$ by $f(x) = 2x \mod 1$.

Let $\Sigma_2 = \varprojlim(S, f)$

Then Σ_2 is the dyadic solenoid and as we said before, the shift homeomorphism is expansive.

Let $\widehat{S}$ be the unit circle with 1 *sticker* in the complex plane and let $\widehat{f} : \widehat{S} \longrightarrow \widehat{S}$ in the following way:

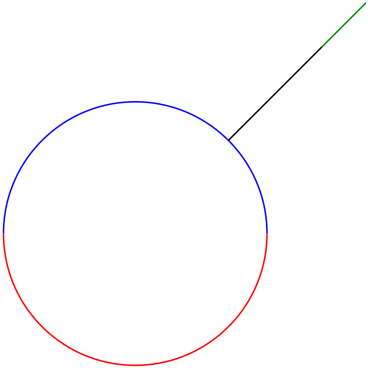
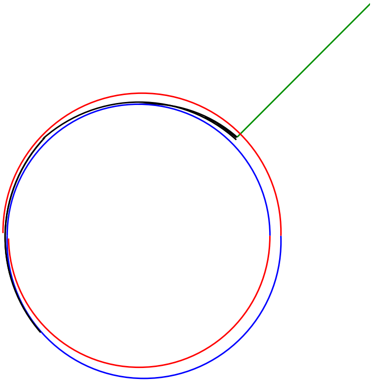


Figure: Doubling and stretch map $\widehat{f}(x)$.



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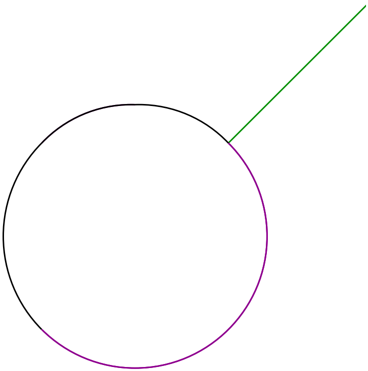

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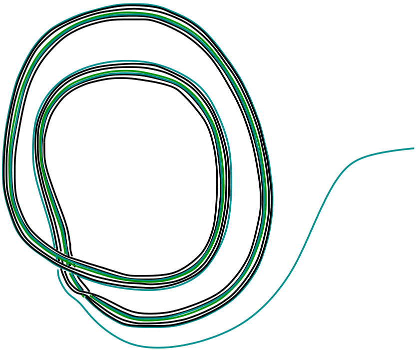

Figure: Inverse limit of $\widehat{f}(x)$ is a ray limiting to the soleniod

Expansive Homeomorphisms of Plane Continua

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Does there exists an expansive homeomorphism of a plane continuum?

Yes, the Plykin Attractor is a one dimensional plane continuum that admits an expansive homeomorphism.

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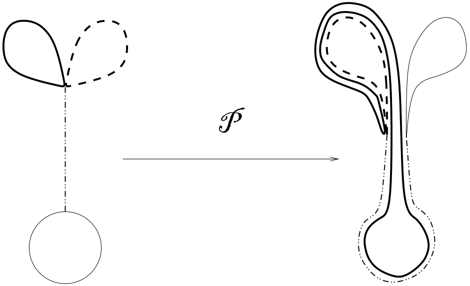

Figure: Plykin attractor admits an expansive homeomorphism

However, the Plykin Attractor is a 1-dimensional 4-separating plane continuum that admits an expansive homeomorphism.

Question

Does there exist an 1-dimensional plane separating continuum that admits an expansive homeomorphism?

No!

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All 1-dimensional non-separating plane continua are tree-like. (Converse is not true.)

Theorem

(M.) Tree-like continua do not admit expansive homeomorphisms.

The proof of this result contains many important ideas and techniques, so it will be valuable to examine it.

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The proof of this result contains many important ideas and techniques, so it will be valuable to examine it.

Let $h : X \longrightarrow X$ be a homeomorphism.

M is an *unstable* subcontinuum of h if $\mathrm{diam}(h^n(M)) \to 0$ as $n \to -\infty$.

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Theorem

(Kato) If $h : X \longrightarrow X$ is an continuum-wise expansive homeomorphism of a continuum, then there exists a stable or unstable subcontinuum.

Since h is expansive if and only if h^{-1} is expansive, we will always assume the existence of an unstable subcontinuum.

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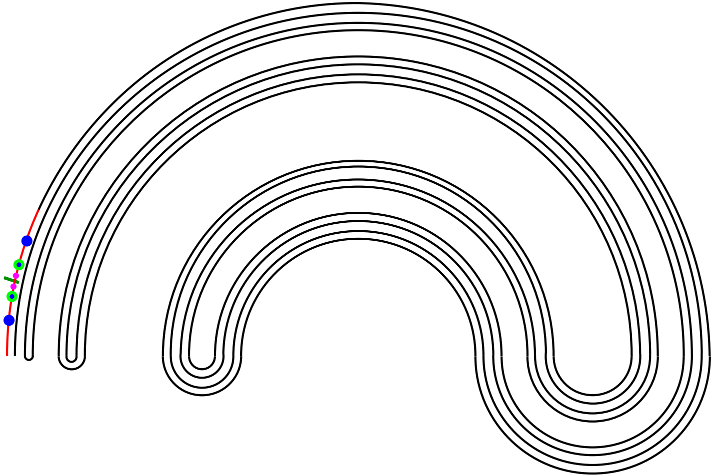

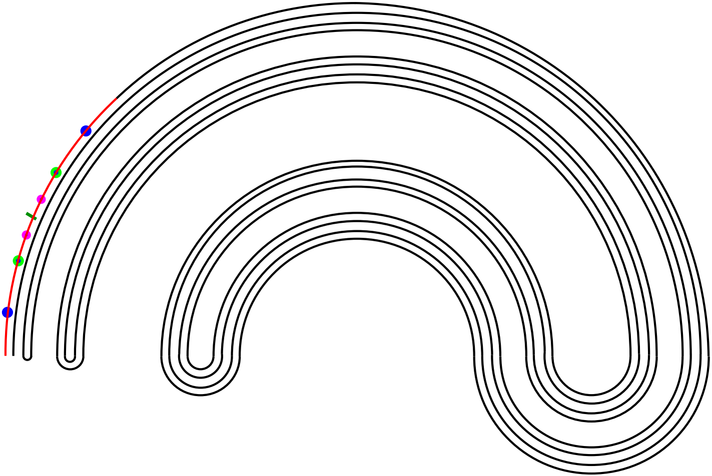

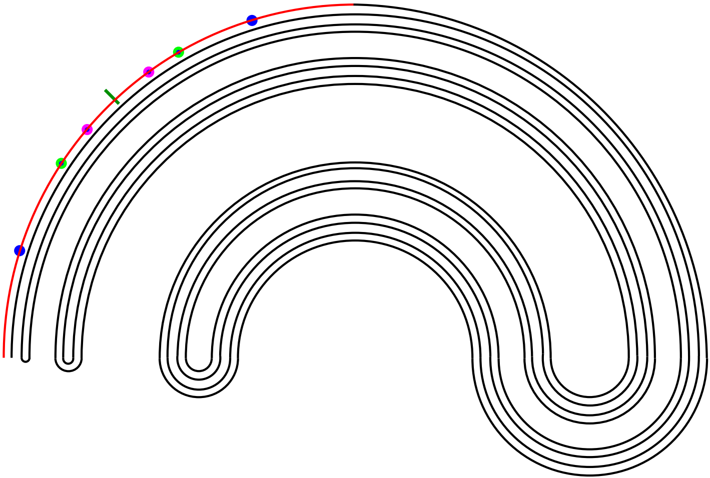

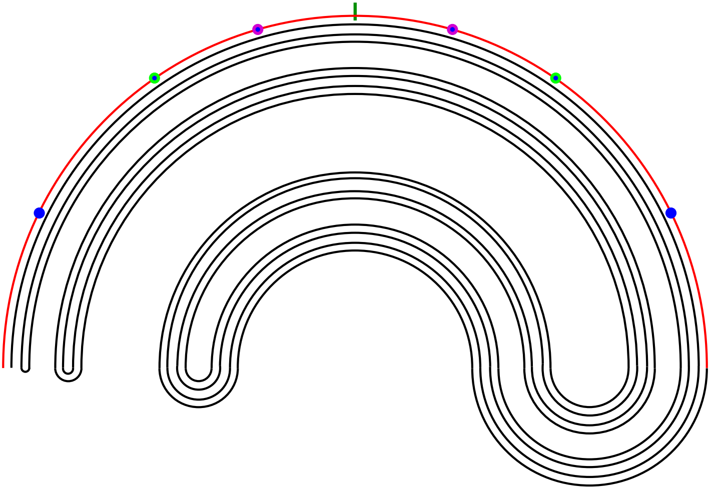

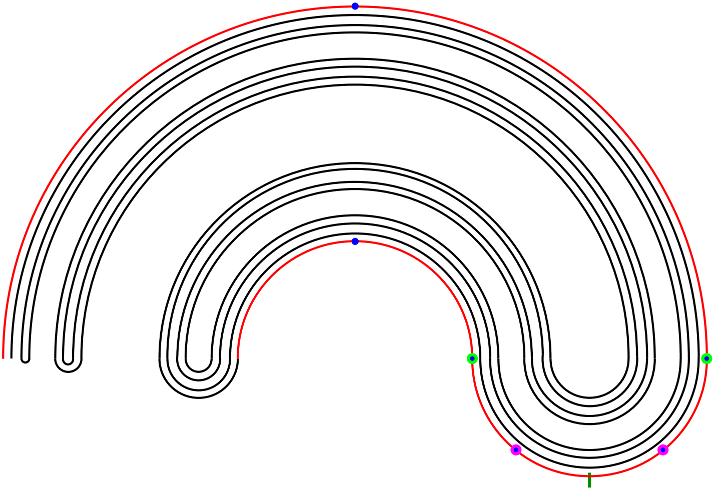

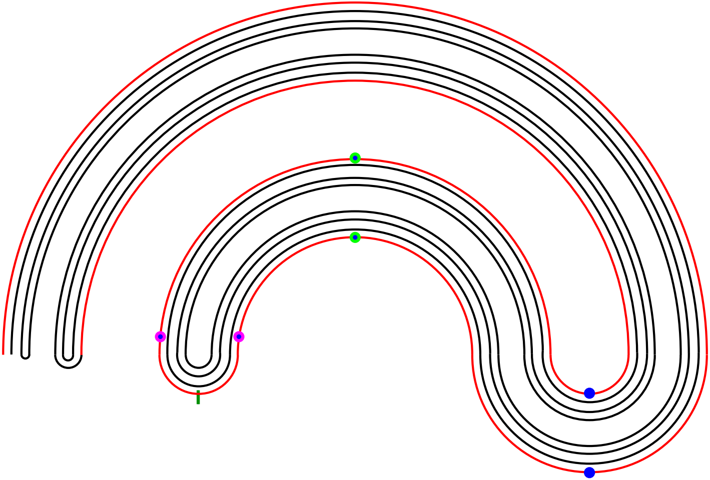

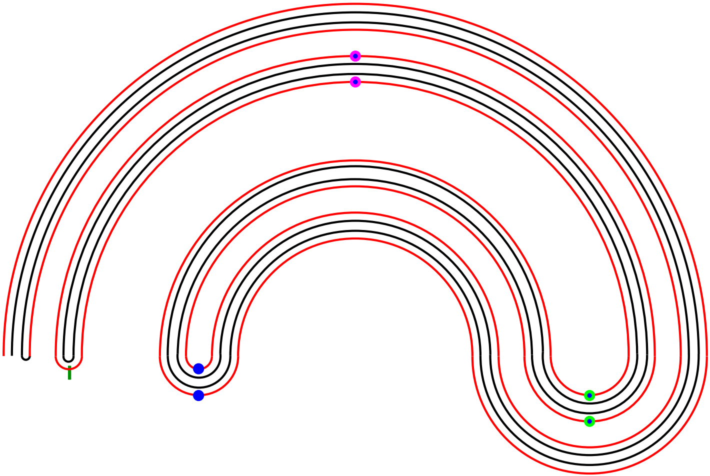

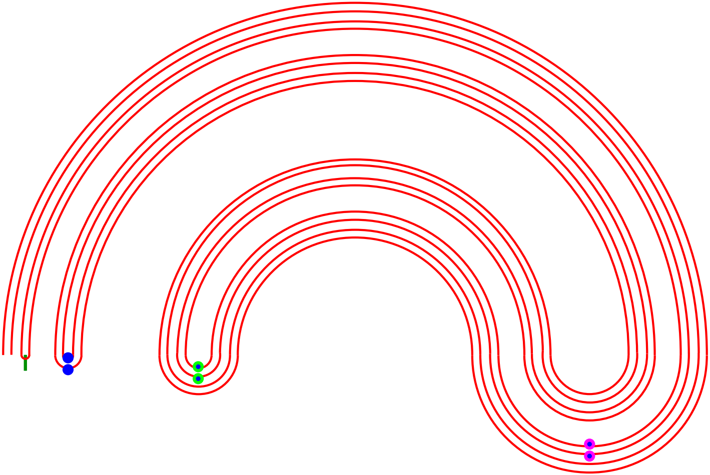

Let $h : X \longrightarrow X$ be a homeomorphism of a continuum X.

Define

$$d_k^n(x, y) = \max\{d(h^i(x), h^i(y)) : k \le i \le n\}.$$

And define

$$d_{-\infty}^n(x, y) = \sup\{d(h^i(x), h^i(y)) : -\infty < i \le n\}.$$

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Lemma

Let $h : X \longrightarrow X$ be a homeomorphism of a compact space X. Suppose that $0 < \epsilon < c$ and for each $n \in \mathbb{N}$ there exists points $x_n, y_n \in X$ such that

$$\epsilon/3 \leq d(x_n, y_n) \text{ and } d^n_{-n}(x_n, y_n) < \epsilon.$$

Then c cannot be an expansive constant.

Proof.

There exist converging subsequences $\{x_{n(i)}\}_{i=1}^{\infty} \to x$ and $\{y_{n(i)}\}_{i=1}^{\infty} \to y$.

Since $\mathrm{d}(x_{n(i)}, y_{n(i)}) \geq \epsilon/3$, x and y must be distinct.

Since $\{n(i)\}_{i=1}^{\infty}$ is strictly increasing, it follows that given $k \in \mathbb{Z}$, then $-n(i) \leq k \leq n(i)$ for all $i \geq |k|$.

So $\mathrm{d}(h^k(x_{n(i)}), h^k(y_{n(i)})) < \epsilon$ for all $i \geq |k|$.

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Suppose

1. $\mathcal{T}$ *is a tree-cover of continuum* X
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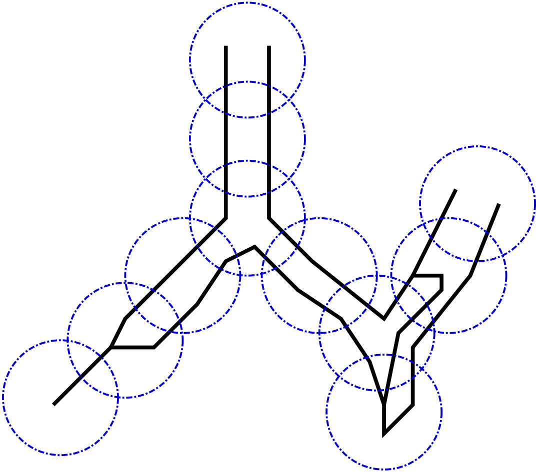


Figure: Tree cover of X and unstable subcontinuum M.

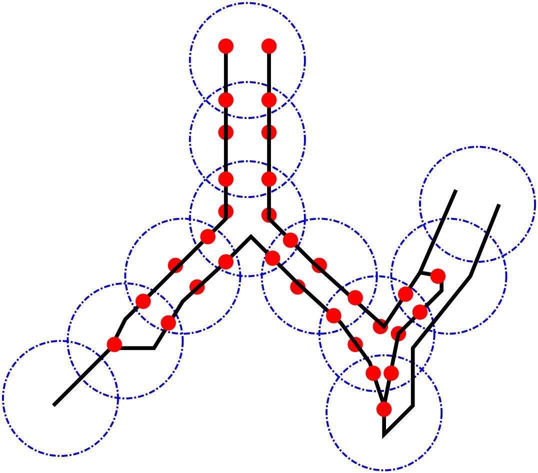


Figure: Simple chain from a to b such that the distance between consecutive points is less that $\epsilon/3$ under d_k^n.

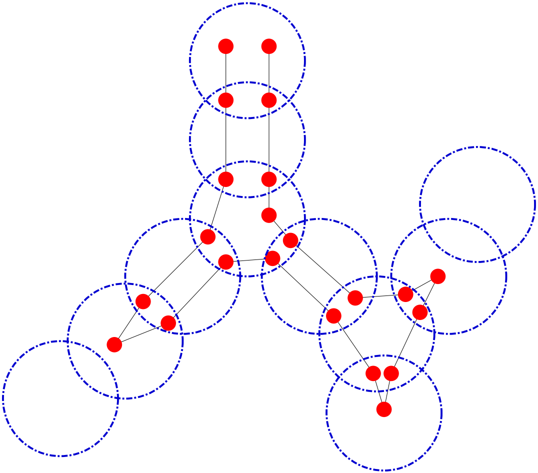
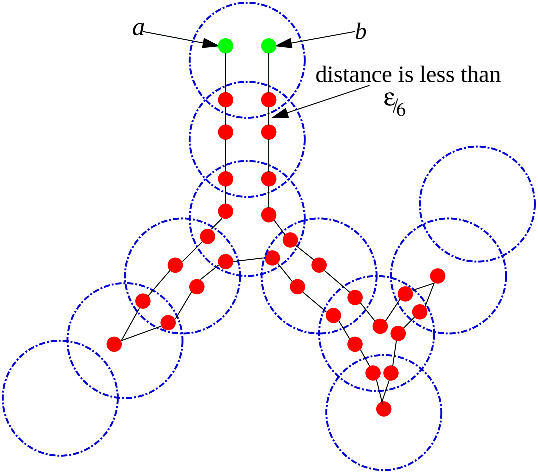


Figure: We only need to consider the simple chain and not the subcontinuum.



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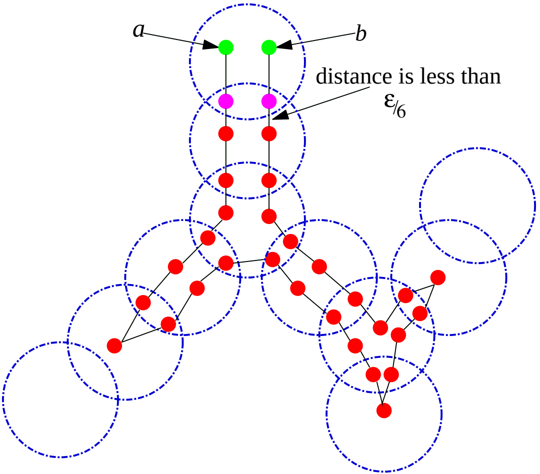

Figure: Hence either $d_k^n(x_\alpha, x_\beta) \geq \epsilon$ or $\epsilon > d_k^n(x_\alpha, x_\beta) \geq \epsilon/3$. If it is the latter, we are done!

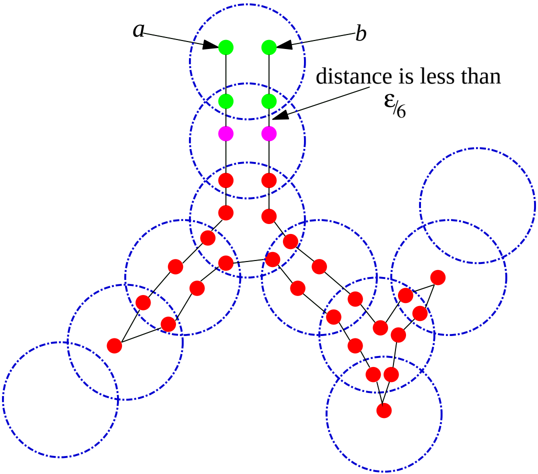

Figure: Hence either $d_k^n(x_{\alpha-1}, x_{\beta+1}) \geq \epsilon$ or $\epsilon > d_k^n(x_{\alpha-1}, x_{\beta+1}) \geq \epsilon/3$. If it is the latter, we are done!

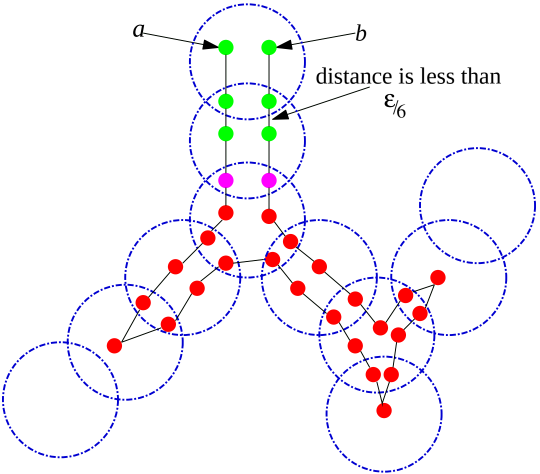

Figure: Hence either $d_k^n(x_{\alpha-2}, x_{\beta+2}) \geq \epsilon$ or $\epsilon > d_k^n(x_{\alpha-2}, x_{\beta+2}) \geq \epsilon/3$. If it is the latter, we are done!

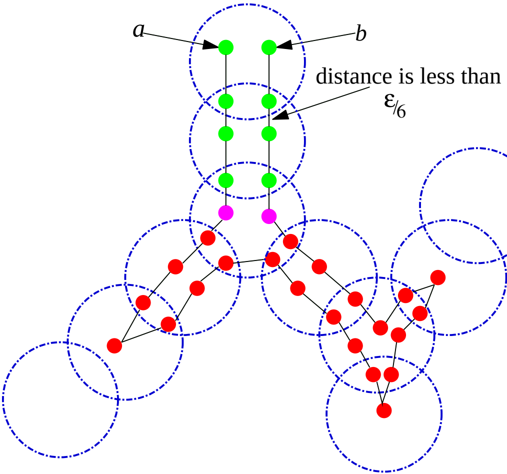

Figure: Hence either $d_k^n(x_{\alpha-3}, x_{\beta+3}) \geq \epsilon$ or $\epsilon > d_k^n(x_{\alpha-3}, x_{\beta+3}) \geq \epsilon/3$. If it is the latter, we are done!

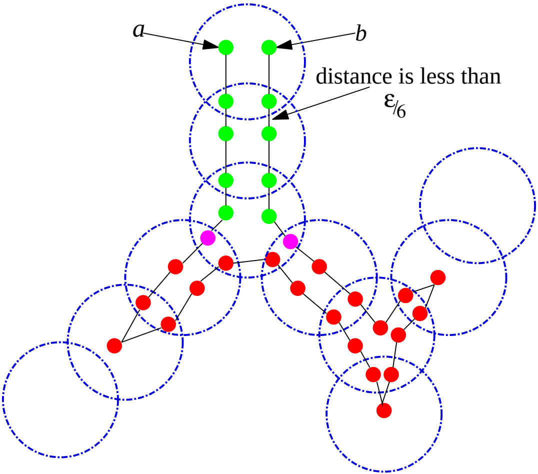

Figure: Hence either $d_k^n(x_{\alpha-4}, x_{\beta+4}) \geq \epsilon$ or $\epsilon > d_k^n(x_{\alpha-4}, x_{\beta+4}) \geq \epsilon/3$. If it is the latter, we are done!

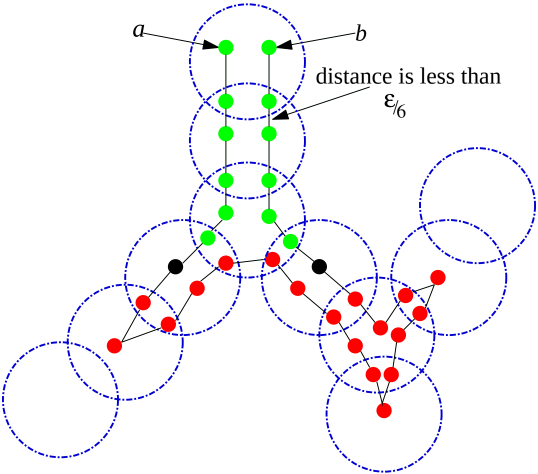

Figure: Hence either $d_k^n(x_{\alpha-5}, x_{\beta+5}) \geq \epsilon$ or $\epsilon > d_k^n(x_{\alpha-5}, x_{\beta+5}) \geq \epsilon/3$. If it is the latter, we are done!

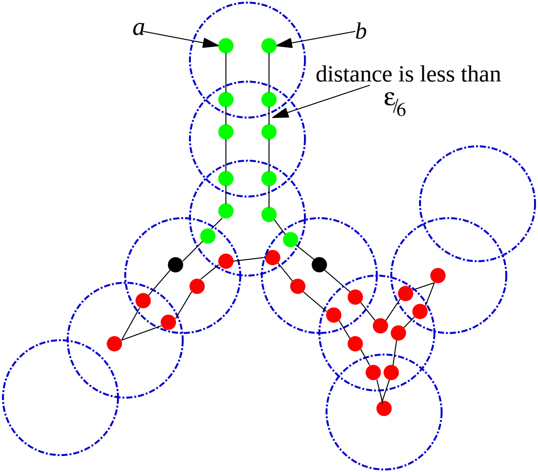

Figure: If this the case we can use the triangle inequality!

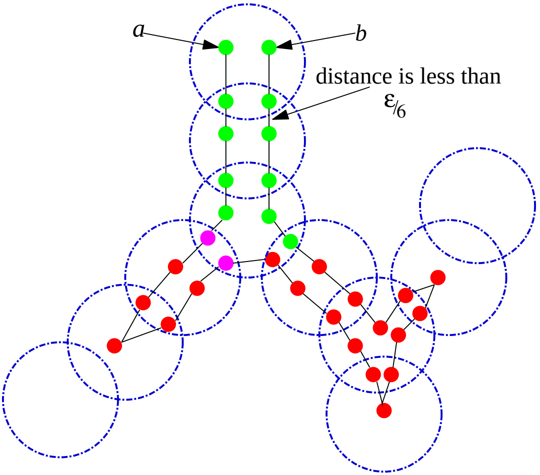

Figure: Hence either $d_k^n(x_{\alpha-6}, x_{\beta+\gamma}) \geq \epsilon$ or $\epsilon > d_k^n(x_{\alpha-6}, x_{\beta+\gamma}) \geq \epsilon/3$. If it is the latter, we are done!

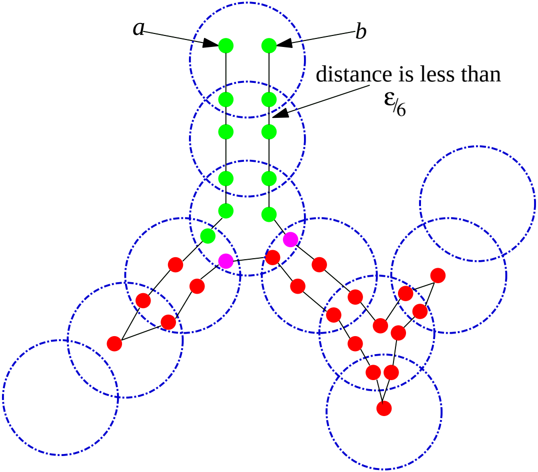

Figure: Hence either $d_k^n(x_{\alpha-7}, x_{\beta+\gamma+1}) \geq \epsilon$ or $\epsilon > d_k^n(x_{\alpha-7}, x_{\beta+\gamma+1}) \geq \epsilon/3$. If it is the latter, we are done!

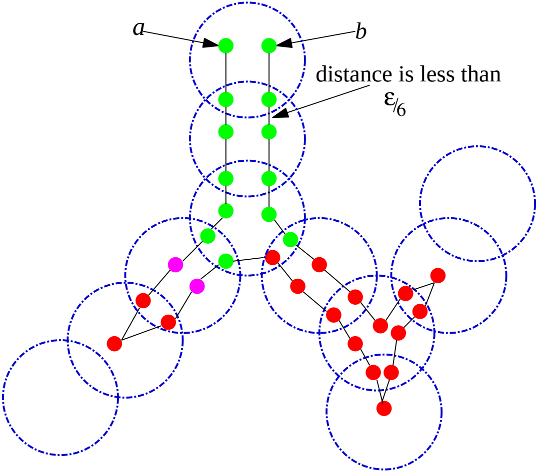

Figure: Hence either $d_k^n(x_{\alpha-8}, x_{\beta+\gamma+2}) \geq \epsilon$ or $\epsilon > d_k^n(x_{\alpha-8}, x_{\beta+\gamma+2}) \geq \epsilon/3$. If it is the latter, we are done!

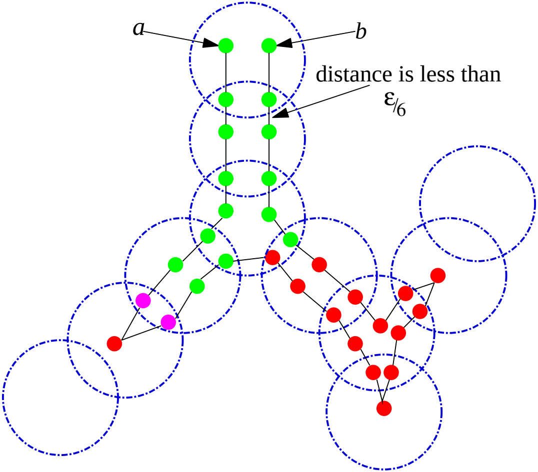

Figure: Hence either $d_k^n(x_{\alpha-9}, x_{\beta+\gamma+3}) \geq \epsilon$ or $\epsilon > d_k^n(x_{\alpha-9}, x_{\beta+\gamma+3}) \geq \epsilon/3$. If it is the latter, we are done!

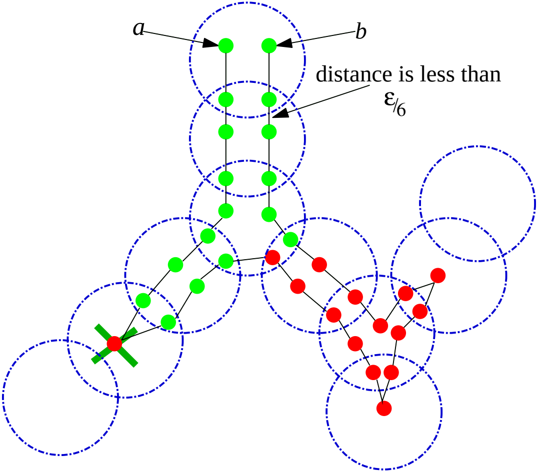

Figure: Hence either $\mathrm{d}_k^n(x_{\alpha-10}, x_{\beta+\gamma+4}) \geq \epsilon$ or $\epsilon > \mathrm{d}_k^n(x_{\alpha-10}, x_{\beta+\gamma+4}) \geq \epsilon/3$. Oops! Contradiction!

Lemma

Suppose

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Then there exists $x_\alpha, x_\beta \in X$ *such that* $\epsilon/3 \leq d_k^n(x_\alpha, x_\beta) < \epsilon$ *and* x_α, x_β *are in the same element of* $\mathcal{T}$*.*

Theorem

Let $h : X \longrightarrow X$ be a homeomorphism and $0 < \epsilon < c$. Suppose that M is an unstable subcontinuum of h such that for every $\delta > 0$ there exist an integer $k = k(\delta)$ and a tree-cover $\mathcal{T}_k$ of $h^k(M)$ with the following properties:

1. $mesh(\mathcal{T}_k) < \delta$
2. *there exist points* $x_k, y_k \in h^k(M)$ *that are in the same element of* $\mathcal{T}_k$
3. $d^0_{-\infty}(x_k, y_k) > \epsilon$.

Then c cannot be an expansive constant for h.

Proof.

Since subcontinua of unstable subcontinua are unstable, we may choose M such that $\text{diam}(h^i(M)) < \epsilon/2$ for all $i \leq 0$.

Choose δ_k such that if $d(x, y) < \delta_k$ then $d(h^i(x), h^i(y)) < \epsilon$ for all $0 \leq i \leq k$.

By (3) and a previous Lemma, there exists $\hat{x}_k, \hat{y}_k \in h^k(M)$ such that $\hat{x}_k, \hat{y}_k$ are in the same element of $\mathcal{T}_k$ and $\epsilon/3 \leq d^0_{-k}(\hat{x}_k, \hat{y}_k) < \epsilon$.

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$$\epsilon/3 \leq d(w_k, z_k) \text{ and } d^k_{-k}(w_k, z_k) < \epsilon.$$

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Corollary

Tree-like continua do not admit expansive homeomorphisms.

Proof.

Suppose on the contrary that $h : X \longrightarrow X$ is expansive with expansive constant c.

Let $0 < \epsilon < c$. We may assume that M is an unstable subcontinuum of h.

Let $\delta > 0$ and $\mathcal{T}_\delta$ be a tree-cover of X with mesh less than δ.

Let A_δ be $|\mathcal{T}_\delta| + 1$ points of M.

Since h is expansive, there exists a k such that $\mathrm{d}^k_\infty(x, y) > c$ for all distinct $x, y \in A_\delta$.

By the pigeon-hole principal there exists distinct $x', y' \in A_\delta$ such that $w = h^k(x')$ and $z = h^k(y')$ are in the same element of $\mathcal{T}_\delta$ (Notice $\mathrm{d}^k_\infty(w, z) > c > \epsilon$).

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The Plykin Attractor is 4-separating and admits an expansive homeomorphism.

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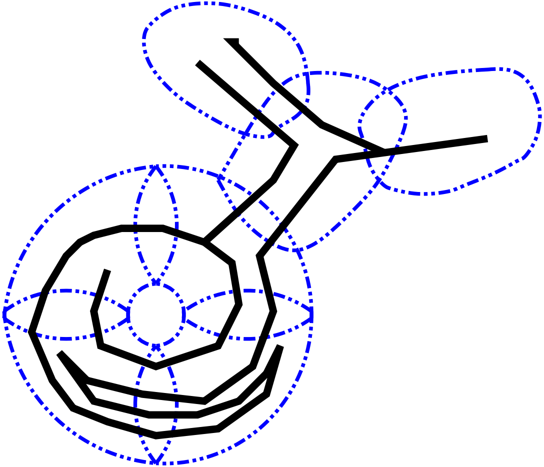
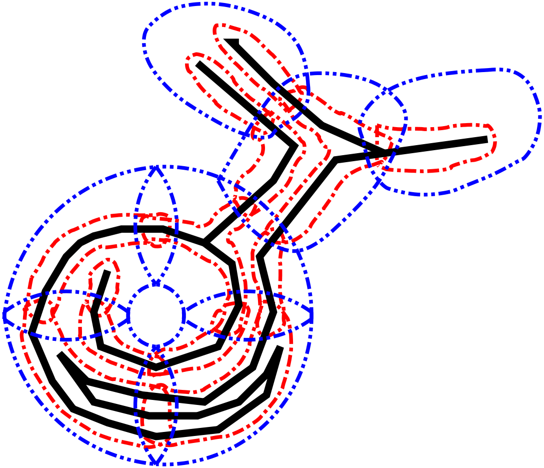


Figure: Let X be a continuum, Y be a tree-like subcontinuum and $\mathcal{U}$ be a finite open cover of X. Then define $T(Y,\mathcal{U})$ to be a tree cover of Y of minimal cardinality that refines $\mathcal{U}$.



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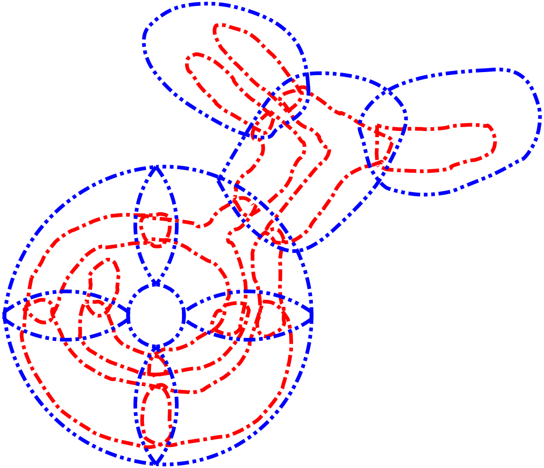

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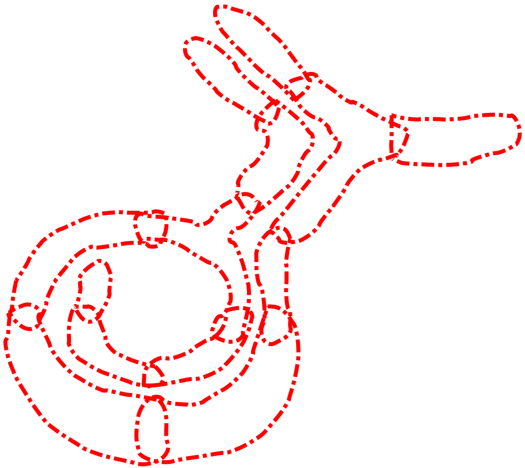

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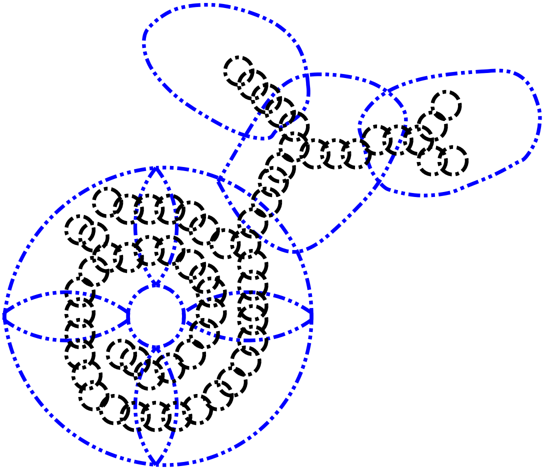
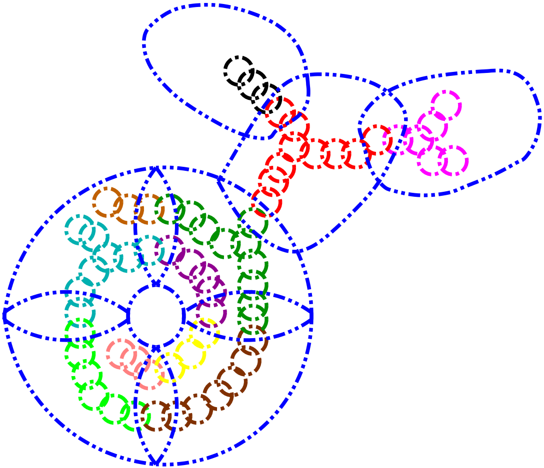


Figure: Likewise, if $\mathcal{T}$ is a tree cover that refines $\mathcal{U}$, then define $T(\mathcal{T},\mathcal{U})$ to be a tree cover of minimal cardinality that refines $\mathcal{U}$ and is refined by $\mathcal{T}$.



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Lemma

Let $h : X \longrightarrow X$ be a homeomorphism, and M be an unstable subcontinuum for h. Suppose that for every $\delta > 0$ there exists

1. *a finite open cover $\mathcal{U}_\delta$ of X*
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2. $|T(h^k(M), \mathcal{U}_\delta)| < |E_\delta|$.

Then h is not an expansive homeomorphism.

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This follows from a similar "pigeon-hole principal" argument as the tree-like result. □

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Combining this with the following theorem:

Theorem

(Kato) Let $h : X \longrightarrow X$ be a continuum-wise expansive homeomorphism and M be an unstable subcontinuum.

Then there exists $p > 1$ and a collection of subsets $\{E_n\}_{n=1}^{\infty}$ of M such that

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That is unstable tree-like subcontinua must wrap (not fold) in the continuum at an exponential rate!

Corollary

Suppose X is a 1-dimensional continuum that separates the plane into 2 complementary domains. Then X does not admit an expansive homeomorphism.

Proof.

First, every unstable subcontinuum M must be tree-like. It can be shown (with much work) that if $\mathcal{U}$ is a finite open cover of X, then $|T(h^k(M), \mathcal{U})|$ has a polynomial growth rate. This is due to the fact that indecomposable 2-separating must be created by more folding than wrapping (as in the solenoid.)

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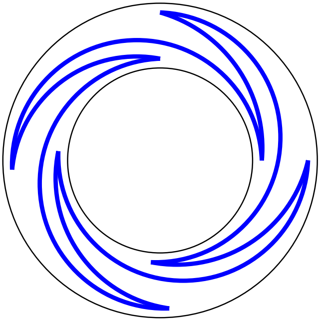

Figure: Although there is some wrapping in a 2-separating plane continuum, it can be shown that there must be "more" bending.

Conjecture 1-dimensional 3-separating plane continua do not admit expansive homeomorphisms.

It appears that there is a "little bit more" bending than wrapping in 1-dimensional 3-separating plane continua.

I have some new techniques to "measure " this bending but currently it is very technical.

However, there is more we can say: Let X be a continuum. Y is a *minimal cyclic* subcontinuum of X if Y is not tree-like but every proper subcontinuum is tree-like.

Theorem

If $h : X \longrightarrow X$ is a expansive homeomorphism of a minimally cyclic continuum X, then h (or h^{-1}) is positively continuum-wise fully expansive.

This follows from the fact that unstable subcontinua must wrap more and more in X. Since X is minimally cyclic, they converge to X in the Hausdorff metric. With a little more work, it can be shown that every subcontinuum has this property.

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Also, if X is a k-cyclic continuum, then it has a finite number of minimally cyclic subcontinua.

Hence if $h : X \to X$ is a homeomorphism and Y is a minimally cyclic subcontinuum, then there exists a k such that $h^k(Y) = Y$.

Note: h^k is expansive if and only if h is expansive. So,

Theorem

Suppose that $h : X \longrightarrow X$ is a expansive homeomorphism of a k-cyclic continuum. Then there exists $k \in \mathbb{N}$ and a subcontinuum Y such that $h^k|_Y : Y \longrightarrow Y$ is fully expansive. Furthermore, Y is indecomposable and minimally cyclic.

A *fully expansive homeomorphism* is one that is expansive and either h or h^{-1} is positively continuum-wise fully expansive.

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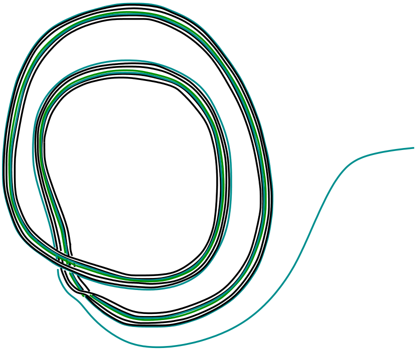

Figure: Here, the solenoid is minimally cyclic and the restriction homeomorphism to the solenoid is fully expansive.

Questions

Question

If $h : X \longrightarrow X$ is a fully expansive homemorphism of k-cyclic continuum, then is X is homeomorphic to the inverse limit of the bouquet of circles?

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If X is a circle-like continuum that admits an expansive homeomorphism, then X is a solenoid formed by the same bonding map.

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Suppose X is a k-cyclic continuum that admits an expansive homeomorphism. Is X the inverse limit of the same graph G and same bonding map $f : G \longrightarrow G$ such that the shift homeomorphism of f is fully expansive?

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Suppose that $h : X \longrightarrow X$ is an expansive homeomorphism of a k-cyclic continuum (or a G-like continuum). Does there exist a graph H and a map $f : H \longrightarrow H$ such that

1. *X is homeomorphic to $Y = \varprojlim(H, f)$*
2. *the shift homeomorphism $\widehat{f}$ of $\varprojlim(H, f)$ is expansive*
3. *there exists a map $\phi : X \longrightarrow Y$ such that $\widehat{f} \circ \phi = \phi \circ h$?*

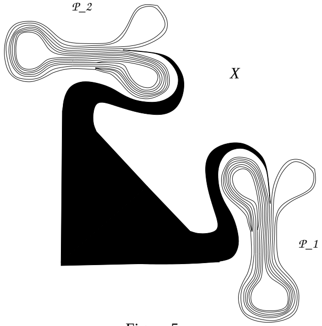

Figure 5

Figure: 2-dimensional plane continuum that admits an expansive homeomorphism

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Does there exist a **2-dimensional non-separating plane continuum** *that admits an expansive homeomorphism?*

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Does there exists a plane continuum that admits an expansive homeomorphism and separates the plane into an **infinite number of complementary domains**.

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A continuum is *hereditarily equivalent* if it is homeomorphic to each of its proper nondegenerate subcontinua. (These are arcs and psuedo-arcs.)

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