On group-valued continuous functions: k-groups and reflexivity

Gábor Lukács

lukacs@topgroups.ca

Halifax, Nova Scotia, Canada

Notations

For $X, Y \in \mathsf{Haus}$ and $G, H \in \mathsf{Ab}(\mathsf{Haus})$:

- $\mathscr{C}(X,Y) :=$ cts functions, with compact-open topology.
- $\mathscr{C}(X,G)$ is a top. group with pointwise operations.
- $\mathscr{H}(G,H) := \mathscr{C}(G,H) \cap \hom(G,H)$.

Put $\mathbb{T} := \mathbb{R}/\mathbb{Z}$.

- $\hat{G} := \mathscr{H}(G, \mathbb{T})$.
- $\alpha_G \colon G \to \hat{\hat{G}}$ is the evaluation homomorphism, $(\alpha_A(g))(\chi) = \chi(g)$.

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- Is α_G surjective?
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- Is α_G open onto its image?
  - α_V is not open onto its image for a non-locally convex topological vector space V.

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Terminology:

- G is **reflexive** if α_G is a topological isomorphism.
- G is **almost reflexive** if α_G is an open isomorphism.

Pontryagin duality for LCA

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  - if H is compact, then $H^{\perp}$ is open in $\hat{L}$.
- $c(L)^{\perp} = B(\hat{L})$ and $B(L)^{\perp} = c(\hat{L})$, where:
  - $c(L) :=$ connected component of 0 in L.
  - $B(L) := \{x \in L \mid \overline{\langle x \rangle}$ is compact$\}$.

Observations and motivation

Let $X \in \mathsf{Haus}$ and $G \in \mathsf{Ab}(\mathsf{Haus})$.

- If α_G is injective, then so is $\alpha_{\mathscr{C}(X,G)}$.
- If α_G is an embedding, then $\alpha_{\mathscr{C}(X,G)}$ is open onto its image (G being LQC implies that $\mathscr{C}(X,G)$ is so).

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Motivation

- Is $\mathscr{C}(X,G)$ (almost) reflexive?
- What does $\widehat{\mathscr{C}(X,G)}$ look like?

Hausdorff k-spaces

Let $X, Y \in \mathsf{Haus}$ and $G \in \mathsf{Ab}(\mathsf{Haus})$.

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- X is a **k-space** if every k-cts map on X is cts.
  - If X is LC or metrizable, then it is a k-space.
  - If X is a k-space and Y is locally compact, then $X \times Y$ is a k-space.
  - If X is a k-space and G is complete, then $\mathscr{C}(X, G)$ is complete.

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- X is **hemicompact** if its family of compact subsets contains a countable cofinal family (cobase).

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- X is a k**-space** if every k-cts map on X is cts.
- X is **hemicompact** if its family of compact subsets contains a countable cofinal family (cobase).
  - If X is hemicompact and G is metrizable, then $\mathscr{C}(X, G)$ is metrizable.

Special cases

Let X be a Tychonoff k-space.

- $\mathscr{C}(X,\mathbb{T})$ is almost reflexive (Außenhofer, 1999).
- $\mathscr{C}(X,\mathbb{R})$ is almost reflexive (because it is a complete locally convex vector space).
- $\mathscr{C}(X,D)$ is almost reflexive for every discrete group D (because it is complete and has a linear topology).

- Thus, $\mathscr{C}(X,G)$ is almost reflexive for every abelian Lie group ($G = \mathbb{R}^n \times \mathbb{T}^k \times D$).

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Let X be a hemicompact k-space.

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- $\mathscr{C}(X,\mathbb{R})$ is reflexive (because it is a complete metrizable locally convex vector space).
- $\mathscr{C}(X,D)$ is reflexive for every discrete group D (because it is complete, metrizable, and has a linear topology).

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Theorems (GL, 2015)

- If X is a Tychonoff k-space and $G \in$ LCA, then:
  - $\mathscr{C}(X,G)$ is almost reflexive;
  - $\widehat{\mathscr{C}(X,G)} \approx \varinjlim \mathscr{C}(\widehat{K,G/C})$, where $K \subseteq X$ is compact and $C \leq G$ is compact such that G/C is a Lie group.

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- If X is a hemicompact k-space, $G \in$ LCA, and G is metrizable, then $\mathscr{C}(X,G)$ is reflexive.
- If X is compact metrizable and zero-dimensional, and $G \in$ LCA, then $\mathscr{C}(X,G)$ is reflexive.

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  - If G is a k-space that is not LC and $\mathscr{C}(G, \mathbb{R})$ is metrizable, then $G \times \mathscr{C}(G, \mathbb{R})$ is a k-group, but not a k-space. [Hint: evaluation is k-cts but not cts.]

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  - If $H \leq G$ is an open subgroup, then H is a k-group $\Longleftrightarrow$ G is a k-group.

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  - If $H \leq G$ is an open subgroup, then H is a k-group $\iff$ G is a k-group.
  - Product of an arbitrary family of k-groups is a k-group.

Theorem (GL, 2016)

If X is a compact Hausdorff space such that $\mathscr{C}(X,\mathbb{T})$ (or equivalently, $\pi^1(X)$) is divisible and G is LCA, then:

- $\mathscr{C}(X,G)$ is a k-group; and
- $\mathscr{C}(X,G)$ is reflexive.

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- If X is compact Hausdoff and zero-dimensional, and $G \in$ LCA, then $\mathscr{C}(X, G)$ is reflexive.

Idea of the proof

Redaction 1

Let $G \in$ LCA and X compact Hausdorff.

- $G \cong \mathbb{R}^n \times H$, where H contains a compact open subgroup.
- $\mathscr{C}(X,G) \cong \mathscr{C}(X,\mathbb{R})^n \times \mathscr{C}(X,H)$.
- $\mathscr{C}(X,G)$ a k-group $\Longleftrightarrow \mathscr{C}(X,H)$ a k-group.

Thus, WLOG, G contains a compact open subgroup.

Redaction 2

Let $G \in$ LCA with a compact open subgroup O, and X compact Hausdorff.

- $\mathscr{C}(X, O)$ is an open subgroup of $\mathscr{C}(X, G)$.
- $\mathscr{C}(X, G)$ is a k-group $\Longleftrightarrow \mathscr{C}(X, O)$ is a k-group.

Thus, WLOG, G is compact.

Redaction 3

Let G be a compact abelian group and X a compact Hausdorff space such that $\mathscr{C}(X,\mathbb{T})$ is divisible.

- $D :=$ the divisible hull of $\hat{G}$.
- $q\colon \mathscr{C}(X,\hat{D}) \longrightarrow \mathscr{C}(X,G)$ is onto.
- It suffices to show that:
  - $\mathscr{C}(X,\hat{D})$ is a k-group; and
  - q is a quotient map.

Special case

If G is compact abelian with $\hat{G}$ divisible and X is compact Hausdorff, then:

- $\hat{G} \cong \bigoplus D_\alpha$, where $D_\alpha \cong \mathbb{Q}$ or $\mathbb{Z}(p^\infty)$ (countable).
- $G \cong \prod \hat{D}_\alpha$.
- $\mathscr{C}(X, G) \cong \prod \mathscr{C}(X, \hat{D}_\alpha)$ is a k-group, because:
    - each $\hat{D}_\alpha$ is metrizable.

Main Lemma

Let

- X be compact Hausdorff such that $\mathbb{C}(X,\mathbb{T})$ is divisible;
- G a compact group; and
- H a zero-dimensional subgroup.

Then

$$q\colon \mathscr{C}(X,G) \longrightarrow \mathscr{C}(X,G/H)$$

is a quotient map.

Applications: Answers to Gabriyelyan's problems

$\mathfrak{F}_0^I(G)$ and $\mathfrak{F}_\infty^I(G)$

Let I be discrete, and put $I_\infty := I \cup \{*\}$ (compactification).

- $\mathfrak{F}_0^I(G) := \{(g_i)_{i\in I} \mid \lim g_i = 0\}$, with the uniform topology.
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- $\mathscr{C}(K, G)$ is reflexive for all compact zero-dimensional K and $G \in$ LCA, and thus so are $\mathfrak{F}_0^I(G)$ and $\mathfrak{F}_\infty^I(G)$.

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- $\mathscr{C}(K, G)$ is reflexive for all compact zero-dimensional K and $G \in$ LCA, and thus so are $\mathfrak{F}_0^I(G)$ and $\mathfrak{F}_\infty^I(G)$.
- If G is metrizable, then $\mathfrak{F}_0^I(G)$ and $\mathfrak{F}_\infty^I(G)$ are reflexive.

Ideas of the proof: almost reflexivity

Limits

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  - G is NSS;
  - G is a Lie group;
  - $G \cong \mathbb{R}^n \times \mathbb{T}^k \times D$, where D is discrete; and
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  - Every open subgroup is dually embedded.
  - Every subgroup of an LCA is dually embedded.

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Let $\{G_\alpha\}_{\alpha \in I}$ be an inverse system of abelian groups (for every $\alpha, \beta \in I$ there is $\gamma \in I$ such that $\gamma \leq \alpha, \beta$). Put:

- $P = \prod G_\alpha$ and $\pi_\alpha\colon P \to G_\alpha$;
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If $\pi_\alpha(G)$ is dually embedded in G_α for every $\alpha \in I$, then:

- G is dually embedded in P and $\varinjlim \hat{G}_\alpha \to \hat{G}$ is onto;
- if each G_α is almost reflexive, then so is G.

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- if each G_α is almost reflexive, then so is G.

Hence, it suffices to show that the image of $\mathscr{C}(X,G)$ is open in $\mathscr{C}(K,G/C)$ for $K\subseteq X$ and $C\leq G$ compact, and G/C NSS.

The Lie algebra and the exponential map

For $G \in \mathsf{Grp}(\mathsf{Haus})$:

- $\mathscr{L}(G) := \mathscr{H}(\mathbb{R}, G)$;
- $\exp_G := \mathrm{ev}_1 : \mathscr{L}(G) \to G$ is a cts homomorphism.

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If X is a k-space, then:

- $\mathscr{L}(\mathscr{C}(X,G)) = \mathscr{H}(\mathbb{R}, \mathscr{C}(X,G)) \cong \mathscr{C}(X, \mathscr{H}(\mathbb{R}, G))$ $= \mathscr{C}(X, \mathscr{L}(G))$ [because $X \times \mathbb{R}$ is a k-space].
- $(\exp_G)_* = \exp_{\mathscr{C}(X,G)} : \mathscr{C}(X, \mathscr{L}(G)) \to \mathscr{C}(X, G)$.

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If K is compact and H is LCA and NSS, then:

- $\exp_H$ is a local homeomorphism; and thus
- $(\exp_H)_* = \exp_{\mathscr{C}(K,H)}$ has an open image.

A commutative diagram

For $X \in \mathrm{kTych}$, $K \subseteq X$ compact, $G \in \mathrm{LCA}$, and $C \leq G$ compact such that G/C is NSS:

$$\begin{tikzcd}
\mathscr{C}(X,\mathscr{L}(G)) \arrow[r, "\mathscr{L}(\pi_C)_*"] \arrow[d, "(\exp_G)_*"'] & \mathscr{C}(X,\mathscr{L}(G/C)) \arrow[r, "R_K^*"] \arrow[d] & \mathscr{C}(K,\mathscr{L}(G/C)) \arrow[d, "(\exp_{G/C})_*"] \\
\mathscr{C}(X,G) \arrow[r, "(\pi_C)_*"] & \mathscr{C}(X,G/C) \arrow[r, "r_K^*"] & \mathscr{C}(K,G/C)
\end{tikzcd}$$

- $(\exp_{G/C})_*$ has an open image;
- R_K^* is onto by Tietze's Theorem, because $\mathscr{L}(G/C) \cong \mathbb{R}^l$;
- $\mathscr{L}(\pi_C)_*$ is onto, because $\mathscr{C}(X,\mathscr{L}(G)) \cong \mathscr{H}(\hat{G},\mathscr{C}(X,\mathbb{R}))$, $\mathscr{C}(X,\mathbb{R})$ is divisible, and $\widehat{G/C} \cong C^\perp$ is open in $\hat{G}$.

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