# Essential spectra of weighted composition operators on $C(K)$.

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July 20, 2016

# Introduction

- The goal of this talk is to outline some connections between essential spectra of a weighted composition operator $T$,

$$(Tf)(x) = w(x)f(\varphi(x)), x \in K, f \in C(K),$$

on the space $C(K)$ of all complex-valued continuous functions on a Hausdorff compact space $K$ and topological properties of the compact space $K$ and the continuous map $\varphi$ of $K$ into itself.

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- Let $X$ be a Banach space over the field of complex numbers $\mathbb{C}$, $T$ be a bounded linear operator on $X$, and $\sigma(T)$ be the spectrum of $T$.
- $\sigma_1(T) = \sigma(T) \setminus \{\xi \in \mathbb{C}$ such that the set $(\xi I - T)X$ is closed in $X$ and either $null(\xi I - T) < \infty$ or $def(\xi I - T) < \infty \}$.

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- Therefore $\sigma_3(T)$ is the **Fredholm** spectrum of $T$ or equivalently the spectrum of $T$ in the Calkin algebra.
- Notice also that $\sigma_3(T) = \sigma_2(T) \cup \sigma_2(T')$.

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- Finally, $\sigma_5(T) = \sigma(T) \setminus \{\xi \in \mathbb{C}$ : there is a component $C$ of the set $\mathbb{C} \setminus \sigma_1(T)$ such that $\xi \in C$ and the intersection of $C$ with the resolvent set of $T$ is not empty $\}$.

# Introduction

- Obviously $\sigma_1(T) \subseteq \sigma_2(T) \subseteq \sigma_3(T) \subseteq \sigma_4(T) \subseteq \sigma_5(T) \subseteq \sigma(T)$ and simple examples show that all the inclusions above can be proper; but the essential spectral radii $\rho_i(T), i = 1, \dots, 5$ are all the same.

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- It is also well known that the sets $\sigma_i(T), i = 1, 2, 3, 4$ are invariant under compact perturbations of $T$ but $\sigma_5(T)$ in general is not.

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- It is also well known that the sets $\sigma_i(T), i = 1, 2, 3, 4$ are invariant under compact perturbations of $T$ but $\sigma_5(T)$ in general is not.
- The description of the spectrum and essential spectra of weighted composition operators on $C(K)$ in the case when the map $\varphi$ is a homeomorphism of $K$ onto itself is comparatively straightforward and can be illustrated by the following diagrams.

# Essential spectra, invertible case.

$$\lambda \notin \sigma(T)$$

**$K_1$**

$\rho(T) < |\lambda|$

**P**

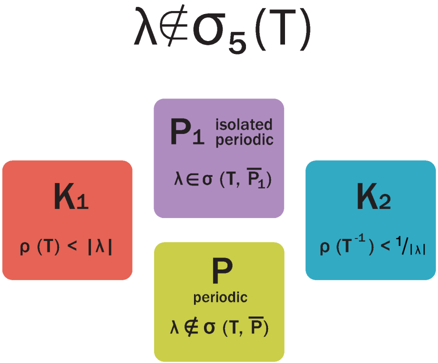
periodic

$\lambda \notin \sigma(T, P)$

**$K_2$**

$\rho(T^{-1}) < 1/|\lambda|$

# Essential spectra, invertible case.



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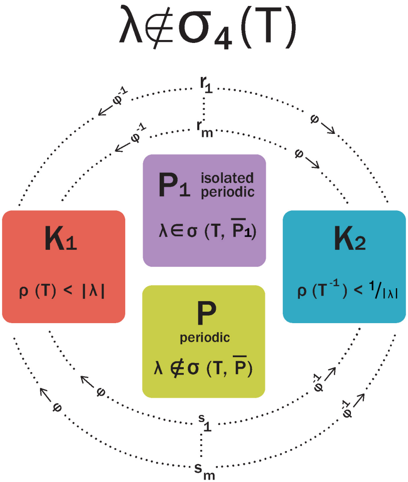

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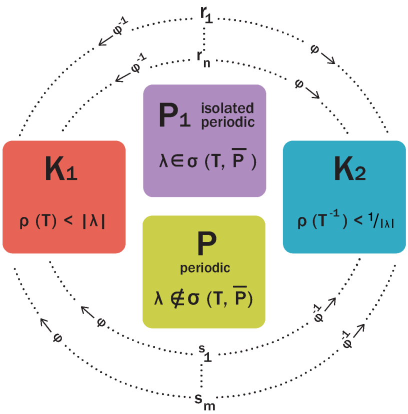
$$\lambda \notin \sigma_3(T)$$



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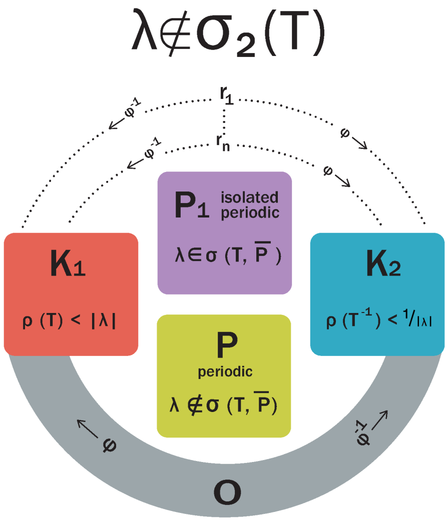

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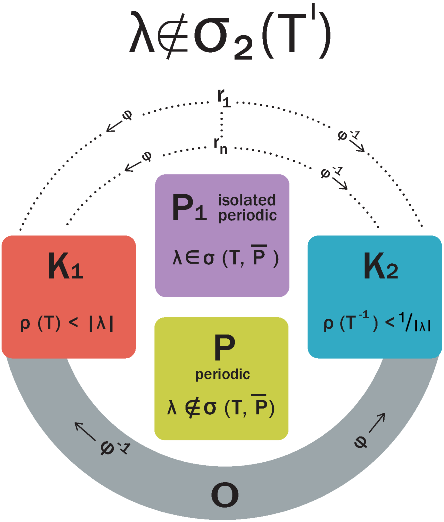

# Essential spectra, non-invertible case.

**Corollary 1**

*(1) If the compact space $K$ has no isolated points then $\sigma(T) = \sigma_3(T)$.*
*(2) If the dynamical system $(K, \varphi)$ is topologically irreducible then $\sigma(T) = \sigma_1(T)$.*

-

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- Let us start with the following seemingly simple question. Let $\varphi$ be a non-invertible surjection of a compact Hausdorff space $K$ onto itself and $T_\varphi$ be the corresponding composition operator on $C(K)$.
- Then $T_\varphi$ is a non-invertible isometry of $C(K)$ into itself and therefore $\sigma(T)$ is the closed unit disk $\bar{D}$.
- It is immediate to see that for any $\lambda \in D$ the operator $\lambda I - T$ is semi-Fredholm, $\ker(\lambda I - T) = \mathbf{0}$, and $ind(\lambda I - T)$ does not depend on $\lambda$.

# Essential spectra, non-invertible case.

- This index will be finite and the operators $\lambda I - T, \lambda \in D$ will be Fredholm if and only if the map $\varphi$ is an **almost homeomorphism** of $K$, i.e. there is a finite subset $F$ of $K$ such that the restriction of $\varphi$ to $K \setminus F$ is a homeomorphism onto $K \setminus \varphi(F)$.

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- For a compact Hausdorff space $K$ it is easily verified that a continuous surjection $\varphi : K \to K$is an almost homeomorphism if and only if there is a finite subset $F$ of $K$ such that the restriction of $\varphi$ to $K \setminus F$ is one-to-one and onto $K \setminus \varphi(F)$.

# Essential spectra, non-invertible case.

- The definition of almost homeomorphism was introduced in our joint paper with Louis Friedler where the following question was raised: what is the class of compact Hausdorff spaces such that every almost homeomorphism of such a space is a homeomorphism? (Shortly we write $K \in AH$). The question in general remains open, but partial results were obtained in our paper with Friedler. Some of these results are as follows ($K$ means an infinite compact Hausdorff space).

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- Assume Continuum Hypothesis, then there is an $F$-space $K$ with countably many isolated points such that $K \in AH$. A surprising and technically quite involved result.
- Let $K$ be arcwise connected and assume one of the following conditions.
  (1) The fundamental group $\Pi_1(K)$ is finite.
  (2) The fundamental group $\Pi_1(K)$ is abelian.
  (3) The fundamental group $\Pi_1(K)$ is finitely generated.
  Then $K \in AH$.

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- Let us turn to **weighted** compositions generated by non-invertible maps.
- The following theorem provides necessary and sufficient conditions for operator $\lambda I - T$ to be surjective (i.e. $\lambda I - T$ is semi-Fredholm and $def(\lambda I - T) = 0$) in the case when the map $\varphi$ is a surjection and the weight $w$ is an invertible element of the algebra $C(K)$.

# Essential spectra, non-invertible case.

- Theorem 2. Let $K$ be a compact Hausdorff space and $\varphi$ be an open continuous map of $K$ onto itself. Let $w$ be an invertible element of $C(K)$. Let $T$ be the weighted composition operator

$$(Tf)(k) = w(k)f(\varphi(k), \ f \in C(K), \ k \in K.$$

Let $\lambda \in \sigma(T)$.

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- (1) $\lambda \neq 0$.
- (2) There is a nonempty open subset $O$ of $K$ such that
  - (a) $\varphi(O) = O = \varphi^{-1}(O)$.
  - (b) For every subset $\{k_n : n \in \mathbb{Z}\}$ of $O$ such that $\varphi(k_n) = k_{n+1}, n \in \mathbb{Z}$ we have

$$(ii)\ \liminf_{n\to\infty} |w_n(k_0)|^{1/n} > |\lambda| \text{ and } \limsup_{n\to\infty} |w_n(k_{-n})|^{1/n} < |\lambda|,$$

    where $w_n = w(w \circ \varphi) \dots (w \circ \varphi^{(n-1)})$.
  - (c) $F = clO \setminus O \neq \emptyset$.
  - (d) $\lambda \notin \sigma(T, C(K \setminus O))$.
  - (e) $\lambda\Gamma \cap \sigma(T, C(F)) = \emptyset$.

# Essential spectra, non-invertible case.

- There are subsets $G$ and $H$ of $F$ with the following properties
  1. $G$ is not empty and $\varphi(G) = G$.
  2. The restriction of $\varphi$ on $G$ is a homeomorphism of $G$ onto itself.
  3. The operator $T_G$ defined on $C(G)$ by the formula $T(f|G) = (Tf)|G$ is invertible and $\rho(T_G^{-1}) < 1/|\lambda|$.
  4. $\exists m \in \mathbb{N}$ such that $G \subset Int_F \varphi^{-m}(G)$.
  5. Let $H = F \setminus \bigcup\limits_{n=1}^{\infty} \varphi^{-n}(G)$. Then $H$ is a closed subset of $F$, $\varphi(H) = H$, and $\rho(T_H) < |\lambda|$.

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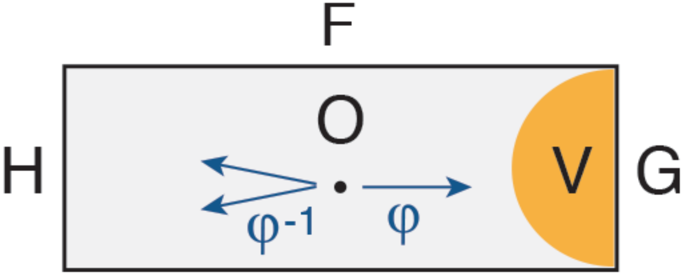
  5. Let $H = F \setminus \bigcup_{n=1}^{\infty} \varphi^{-n}(G)$. Then $H$ is a closed subset of $F$, $\varphi(H) = H$, and $\rho(T_H) < |\lambda|$.
- There are an open neighborhood $V$ of $G$ in $clO$ and $m \in \mathbb{N}$ such that $V \cap H = \emptyset$, $\varphi(V) \subset V$, $clO \setminus \bigcup_{n=1}^{\infty} \varphi^{-n}(V) = H$, the map $\varphi : V \to \varphi(V)$ is a homeomorphism, and $|w_m| > 1$ on $clV$.

# Essential spectra, invertible case.



# Essential spectra, non-invertible case.

**Corollary 2**

*Let $K$ be a compact Hausdorff space and $\varphi$ be a continuous surjection of $K$ onto itself. Assume that at least one of the following conditions is satisfied.*
*(1) For any closed nonempty subset $G$ of $K$ such that $\varphi(G) = G$ the restriction of $\varphi$ onto $G$ is not a homeomorphism.*
*(2) For any open nonempty subset $V$ of $K$ such that $\varphi(V) \subseteq V$ the restriction of $\varphi$ onto $V$ is not a homeomorphism.*
*Let $w \in C(K)^{-1}$ and $T = wT_\varphi$ be the corresponding weighted composition operator.*
*Then for any $\lambda \in \sigma(T)$ we have $(\lambda I - T)C(K) \neq C(K)$.*

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- where $|\zeta| = 1$ and $|a_i| < 1, i = 1, \dots, n$.
- A finite Blaschke product $B$ is called **hyperbolic** if for some $m \in \mathbb{N}$ we have $|(B^{(m)})'| > 1$ on $\partial D$.

# Essential spectra, non-invertible case.

- Proposition. Let $B$ be a finite hyperbolic Blaschke product ($n \geq 2$) and let $w$ be an invertible element of $C(\partial D)$. Let $T = wT_B$ be the corresponding weighted composition operator on $C(\partial D)$. Then

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- (2) $\sigma(T) = \rho\bar{D}$.

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- (1) The spectral radius $\rho = \rho(T)$ is $> 0$.
- (2) $\sigma(T) = \rho\bar{D}$.
- (3) For any $\lambda \in \mathbb{C}$ such that $|\lambda| < \rho$ the operator $\lambda I - T$ is semi-Fredholm, $\ker(\lambda I - T) = \mathbf{0}$, and $ind(\lambda I - T) = \infty$

# Essential spectra, non-invertible case.

- Corollary. Assume conditions of the previous proposition and assume additionally that $w$ is an element of disk algebra $A(D)$ of all functions analytic in $D$ and continuous in $\bar{D}$. Consider operator $T = wT_{\varphi}$ on $A(D)$. Then the statements (1) - (3) of the previous proposition remain valid.