Fixed point theorems for maps with various local contraction properties

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Abstract

Let $\langle X, d\rangle$ be a metric space. We compare ten classes of continuous self-maps $f : X \to X$. All of these self-maps are proved to have fixed or periodic points for spaces X with certain topological properties. We will assume X to be

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3. complete and rectifiably path connected
4. complete and d-convex
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The Classics

Definition (#1)

A function $f : X \to X$ is called Contractive, (C), if there exists a constant $0 \leq \lambda < 1$ such that for any two elements $x, y \in X$ we have $d(f(x), f(y)) \leq \lambda d(x, y)$.

Theorem (Banach, 1922)

If (X, d) is a complete metric space and $f : X \to X$ is (C), then f has a unique fixed point, that is, there exists a unique $\xi \in X$ such that $f(\xi) = \xi$.

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Definition (#2)

A function $f : X \to X$ is called Shrinking, (S), if for any two elements $x, y \in X, x \neq y$ we have $d(f(x), f(y)) < d(x, y)$.

Theorem (Edelstein, 1962)

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A function $f : X \to X$ is called Locally Shrinking, (LS), if for any element $z \in X$ there exists an $\varepsilon_z > 0$ such that $f \restriction B(z, \varepsilon)$ is shrinking, i.e. for any two $x \neq y \in B(z, \varepsilon_z)$ we have $d(f(x), f(y)) < d(x, y)$.

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Let $\langle X, d \rangle$ be compact and let $f : X \to X$.

(i) *If f is (LS), then f has a periodic point.* ♠

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A function $f : X \to X$ is called uniformly Pointwise Contracting, (uPC), if there exists a $\lambda \in [0, 1)$ such that for every $z \in X$ there exists an $\varepsilon_z > 0$ such that for any element $x \in B(z, \varepsilon_z)$ we have $d(f(x), f(z)) \leq \lambda d(x, z)$.

Theorem (Hu and Kirk, 1978; proof corrected by Jungck, 1982)

If $\langle X, d\rangle$ is a rectifiably path connected complete metric space and a map $f : X \to X$ is (uPC), then f has a unique fixed point.

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Classics/Recent

Definition (#6)

A function $f : X \to X$ is called Uniformly Locally Contracting, (ULC), if there exist a $\lambda \in [0, 1)$ and an $\varepsilon > 0$ such that for every $z \in X$ the restriction $f \restriction B(z, \varepsilon)$ is contractive with the same $\lambda_z = \lambda$.

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Assume that $\langle X, d\rangle$ is complete and that $f : X \to X$ is (ULC)

- (i) (Edelstein, 1961) *If X is connected, then f has a unique fixed point.*
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Theorem (C & J, Top. and its App. 204 2016 70-78)

Assume that $\langle X, d\rangle$ is compact and rectifiably path connected. If $f : X \to X$ is (PC), then f has a unique fixed point.

Example (C & J, J. Math. Anal. Appl. 434 2016 1267 - 1280)

There exists a Cantor set $\mathfrak{X} \subset \mathbb{R}$ and a (PC) self-map $\mathfrak{f} : \mathfrak{X} \to \mathfrak{X}$ without periodic points.

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The Ten Contracting/Shrinking Properties

Global Properties. $f : X \to X$ is

(C) contractive if

$$\exists\lambda \in [0,1)\forall x, y \in X\,(d(f(x), f(y)) \leq \lambda d(x, y))\,,$$

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Clearly (C) $\implies$ (S).

Each global property gives rise to two kinds of local properties, named local and pointwise, as follows:

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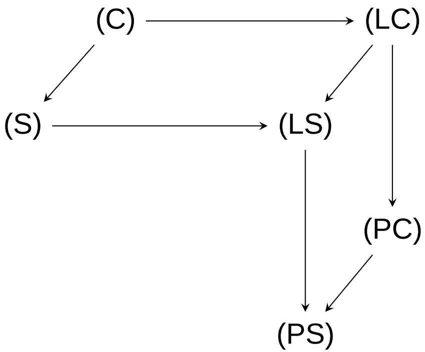
The following implications follow from the definitions:



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(LC) f is *locally contractive if* $\forall z \in X \exists \lambda_z \in [0,1) \exists \varepsilon_z > 0 \forall x, y \in B(z, \varepsilon_z)\, (d(f(x), f(y)) \leq \lambda_z d(x,y))$,

(uLC) f *is (weakly) uniformly locally contractive if* $\exists \lambda \in [0,1) \forall z \in X \exists \varepsilon_z > 0 \forall x, y \in B(z, \varepsilon_z)\, (d(f(x), f(y)) \leq \lambda d(x,y))$,

(ULC) f *is (strongly) Uniformly locally contractive if* $\exists \lambda \in [0,1) \exists \varepsilon > 0 \forall z \in X \forall x, y \in B(z, \varepsilon)\, (d(f(x), f(y)) \leq \lambda d(x,y))$,

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(ULS) f *is Uniformly locally shrinking if* $\exists \varepsilon > 0 \forall z \in X \forall x, y \in B(z, \varepsilon)\, (d(f(x), f(y)) < d(x,y))$.

The Ten Contracting/Shrinking Properties

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The Ten Contracting/Shrinking Properties

Similarly, pointwise properties can be made stronger by requiring uniformity, i.e. that the same $\lambda \in [0,1)$ and/or the same $\varepsilon > 0$ works for all $z \in X$.

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The Ten Contracting/Shrinking Properties or is it 12?

The following implications follow from the definitions:

Remark: (ULS)=(UPS) and (ULC)=(UPC). Any (λ, ε)-(UPC) function is $(\lambda, \frac{\varepsilon}{2})$-(ULC) and (ε)-(UPS) is $(\frac{\varepsilon}{2})$-(ULS).

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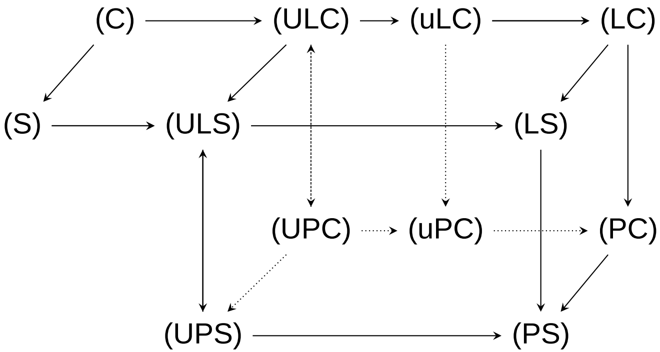

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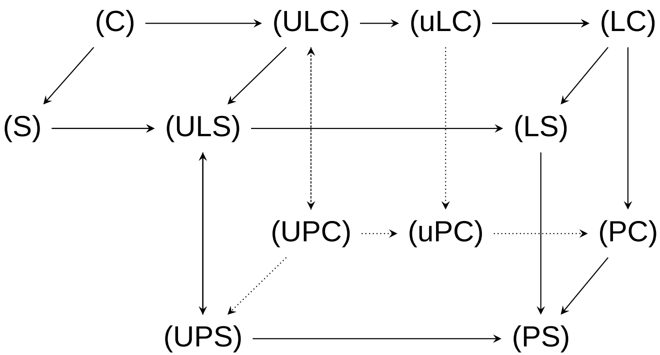

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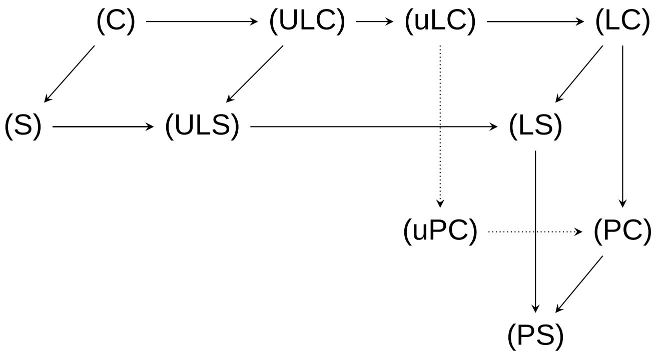
The Ten Contracting/Shrinking Properties

The following diagram

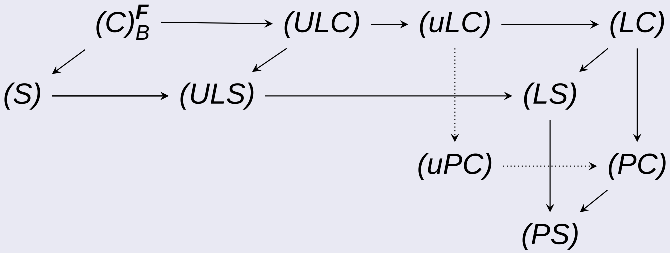


shows the essential classes and implications.

Fixed and Periodic Points

Theorem (Complete Spaces)

Assume X is complete. No combination of any of the properties shown imply any other property, unless the graph forces such implication. Neither does any combination of them imply the existence of a periodic point unless it contains (C).



Fixed and Periodic Points

Theorem (Complete Spaces cont.)

Specifically, there exist 9 *complete spaces X with self-maps $f : X \to X$ without periodic points witnessing the following:*

(PC): $(PC) \not\Leftarrow (S)$

(uPC): $(uPC) \not\Leftarrow (S)\&(LC)$

(LS): $(LS) \not\Leftarrow (uPC)$

(ULS): $(ULS) \not\Leftarrow (uLC)$

(S): $(S) \not\Leftarrow (ULC)$

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Fixed and Periodic Points, Blue does not imply yellow

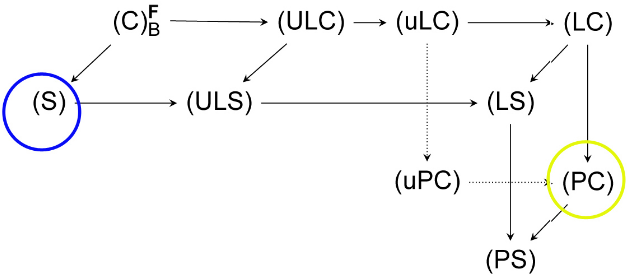

Figure: (PC) $\not\Leftarrow$ (S).

Remark: f is (PC) iff $limsup_{x \to z} \frac{d(f(x),f(z))}{d(x,z)} < 1$ for all $z \in X$.

Take $X = [0, \infty)$ and $f(x) = x + e^{-x^2}$ so $f'(x) = 1 - 2xe^{-x^2}$. We have $f'(0) = 1$ so not-(PC) at $z = 0$. Also $f'[(0, \infty)] \subseteq (0, 1)$ so f is (S) by the MVT. For all $x \in [0, \infty)$,$f(x) > x$ so no periodic points.

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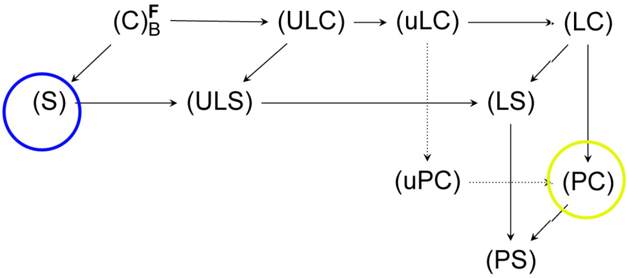

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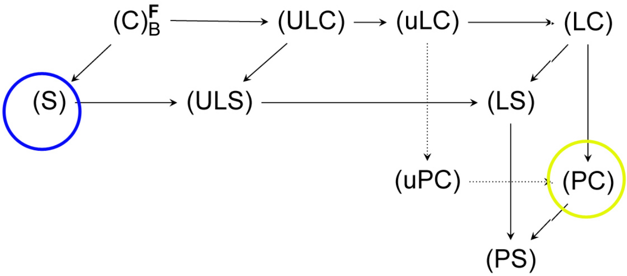

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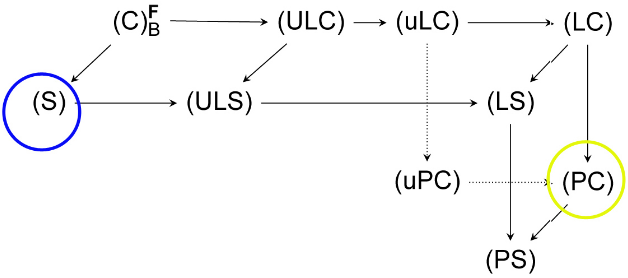

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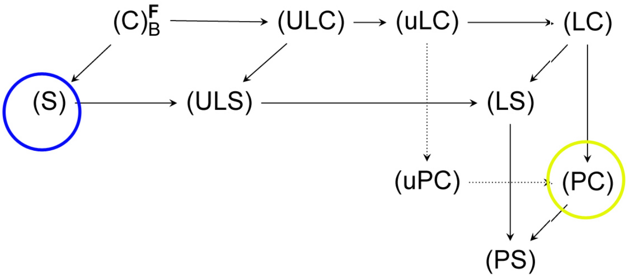

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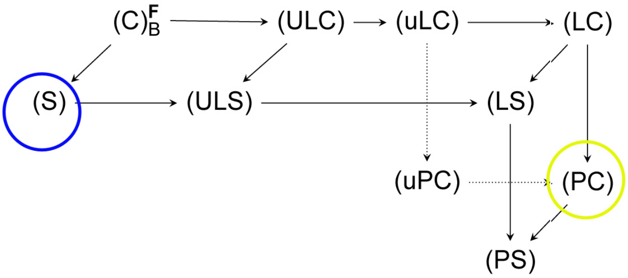

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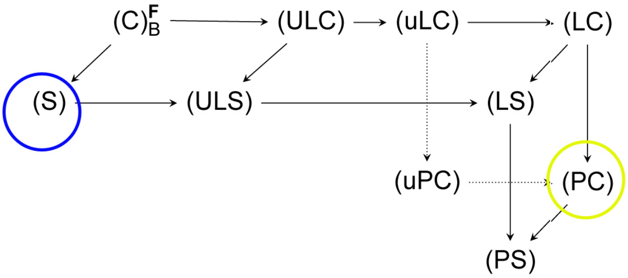

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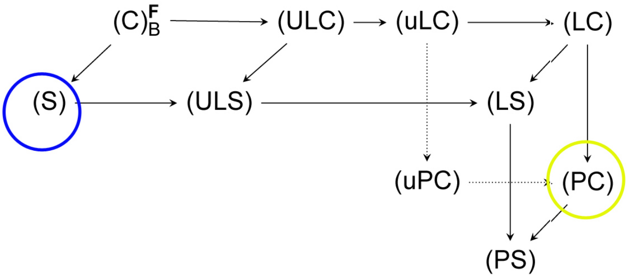

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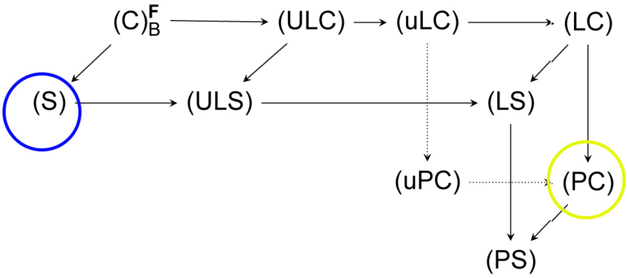

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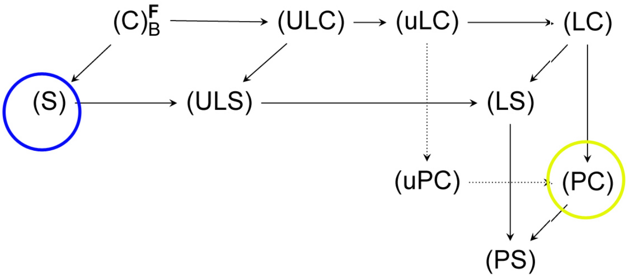

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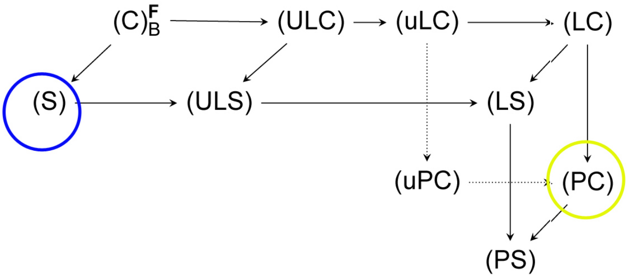

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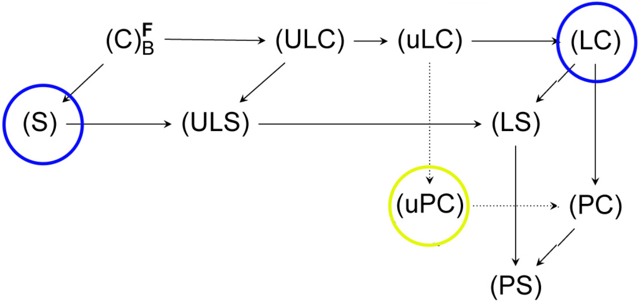

Figure: $(uPC) \not\Leftarrow (S)\&(LC)$. Take $X = \mathbb{R}$ and $f(x) = \frac{1}{2}\left(x + \sqrt{x^2+1}\right)$.

Then $f'(x) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$ so for any $a \in \mathbb{R}$, $f'[(-\infty, a]] = (0, c]$ for some $c < 1$ so MVT gives $(S)\&(LC)$. $\lim_{x\to\infty} f'(x) = 1$ so $\neg(uPC)$.

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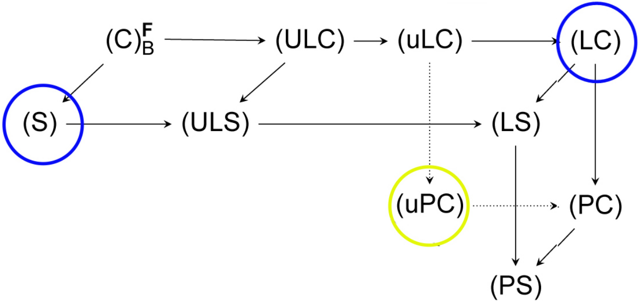

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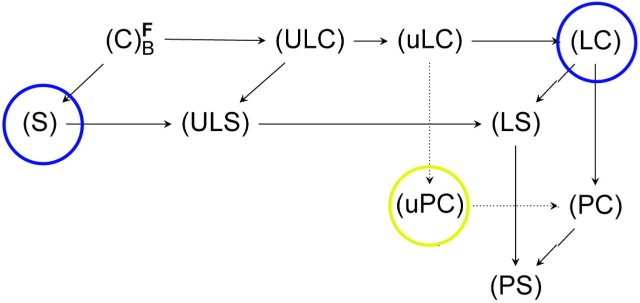

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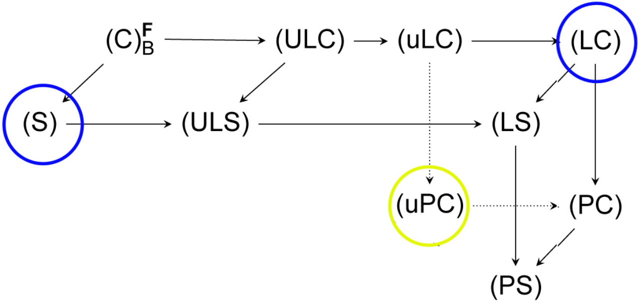

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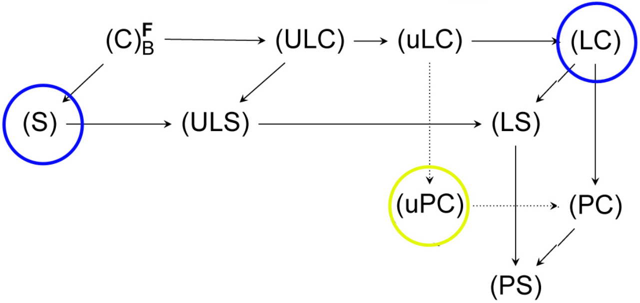
Fixed and Periodic Points, Blue does not imply yellow



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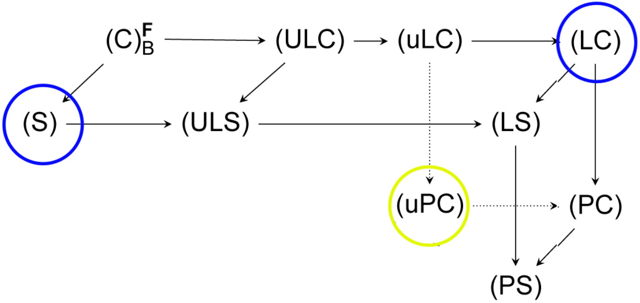

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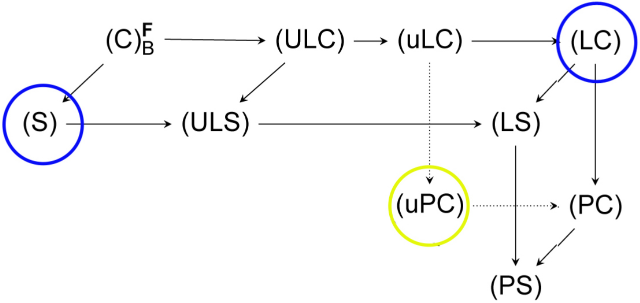

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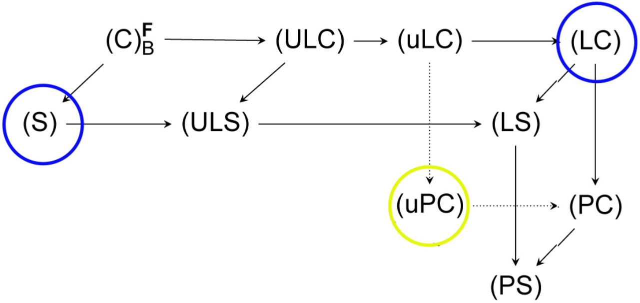

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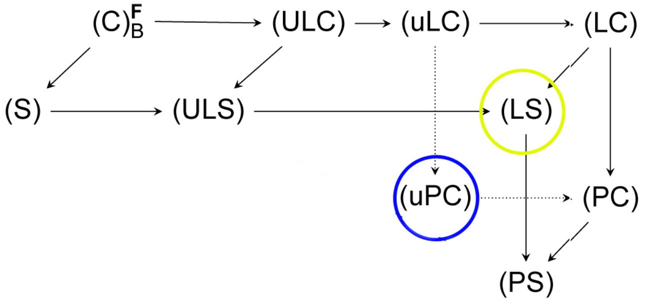

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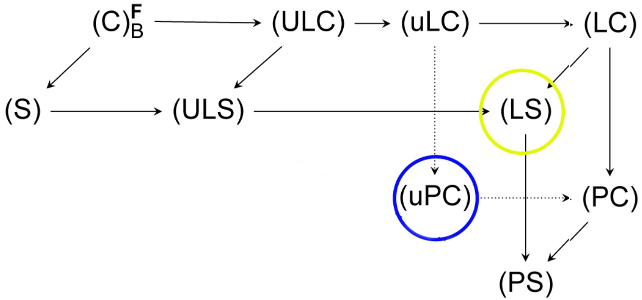

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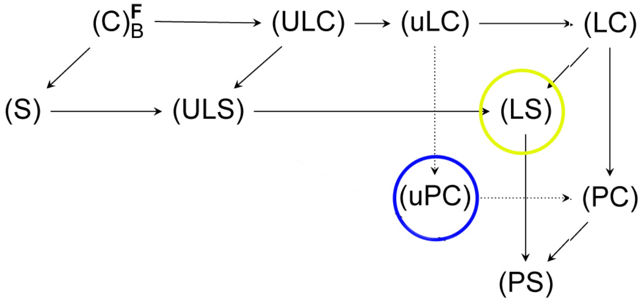

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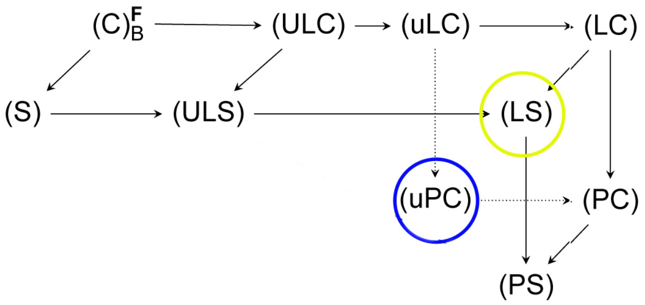

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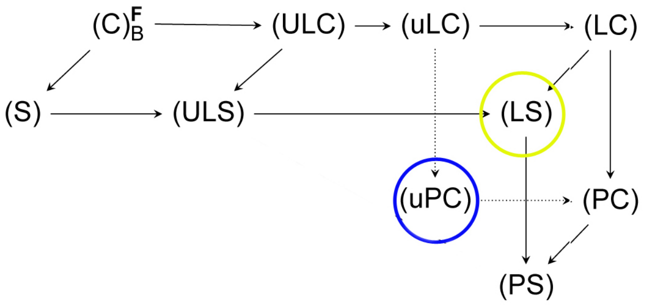

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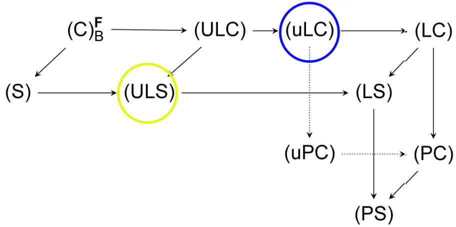

Figure: (ULS) $\not\Leftarrow$ (uLC) Take two increasing sequences: $0 < \beta_n \nearrow 1$ and $0 = a_0 < a_1 < \ldots \nearrow \infty$, $I_n = [a_n, a_{n+1}]$, such that $|I_{2n}| = |I_{2n+1}| = \frac{1}{n+1}$. Define metrics $\rho_n(x,y) = |I_n| \left(\frac{|x-y|}{|I_n|}\right)^{\beta_n}$ on I_n and *"make"* a metric ρ on $X = \bigcup_{n<\omega} I_n$ so that $f : X \to X$, mapping linearly and increasingly I_n onto I_{n+1} has needed properties. For $x \le y$, $n < m$

$$\rho(x,y) = \begin{cases} \rho_n(x,y) & \text{if } x,y \in I_n \\ \rho_n(x,a_{n+1}) + |a_m - a_{n+1}| + \rho_m(a_m,y), & \text{if } x \in I_n, y \in I_m \end{cases}$$

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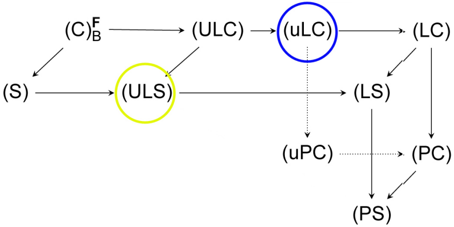

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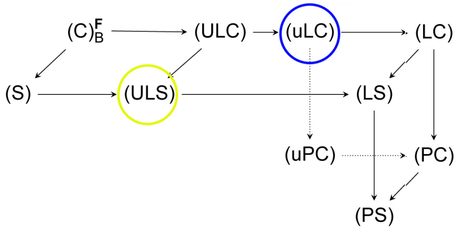

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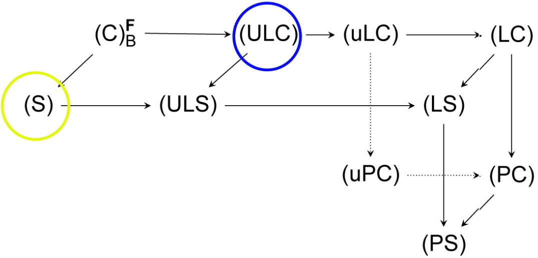

Figure: (S) $\not\Leftarrow$ (ULC) Remetrization.

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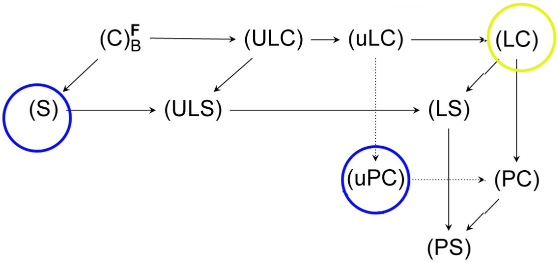

Figure: (LC) $\not\Leftarrow$ (S)&(uPC) Remetrization.

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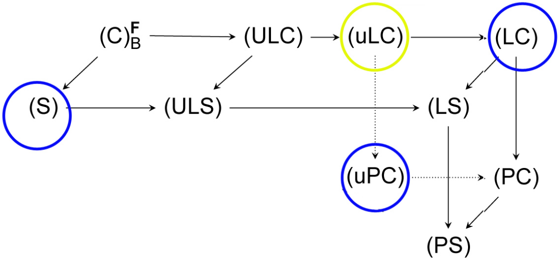

Figure: (uLC) $\not\Leftarrow$ (S)&(LC)&(uPC) Remetrization.

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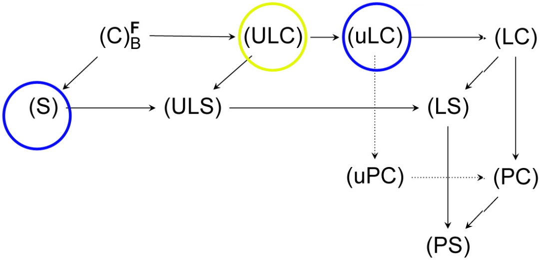

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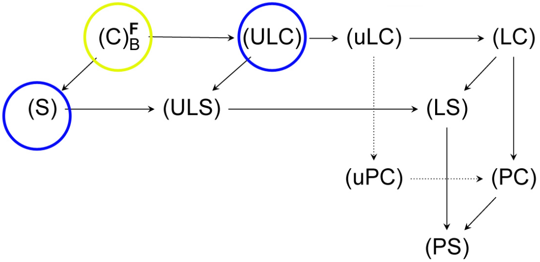
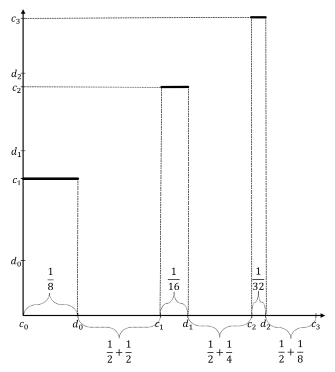


Figure: $(C) \not\Leftarrow (S)\&(ULC)$ We have the following ...

Fixed and Periodic Points, Blue does not imply yellow

Example (A (S)&(ULC)&not(C) map f without periodic points)

Define sequences $\langle c_n \rangle$ and $\langle d_n \rangle$: $c_0 = 0$, $d_n = c_n + 2^{-(n+3)}$ and $c_{n+1} = d_n + \frac{1}{2} + 2^{-(n+1)}$. Set $X = \bigcup_{n<\omega}[c_n, d_n]$ and let $f : X \to X$, $f(x) = c_{n+1}$ for $x \in [c_n, d_n]$. We have



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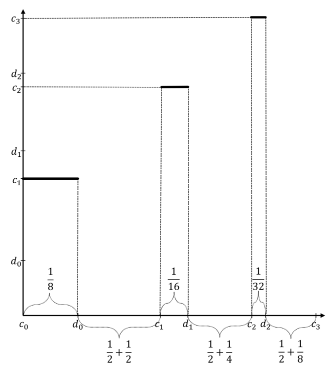

Fixed and Periodic Points - Connected Spaces

Theorem (Connected Spaces)

Assume X is complete and connected. No combination of any of the properties shown imply any other property, unless the graph forces such implication. Neither does any combination imply the exitance of a periodic point unless it contains (C) *or* (ULC).

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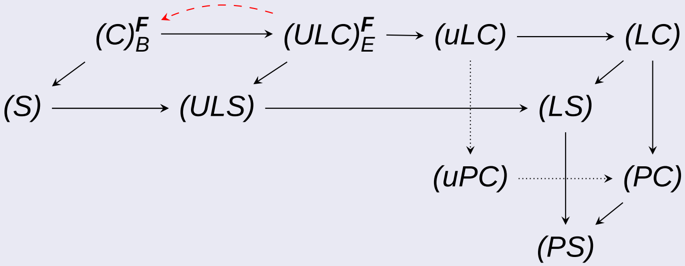

Fixed and Periodic Points - Connected Spaces

A sequence $s = \langle x_0, x_1, ..., x_n \rangle \in X^{n+1}$ is an ε-chain between x_0 and x_n if $d(x_i, x_{i+1}) \leq \varepsilon$. Let $\mathfrak{l}(s) = \sum_{i<n} d(x_i, x_{i+1})$. Define $\hat{D}: X^2 \to [0, \infty), \hat{D}(x, y) = \inf\{\mathfrak{l}(s) : s \text{ is an } \varepsilon\text{-chain between } x \text{ and } y\}$.

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Assume $\langle X, d \rangle$ is connected.

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Fixed and Periodic Points - Connected Spaces

A sequence $s = \langle x_0, x_1, \ldots, x_n\rangle \in X^{n+1}$ is an ε-chain between x_0 and x_n if $d(x_i, x_{i+1}) \leq \varepsilon$. Let $\mathfrak{l}(s) = \sum_{i<n} d(x_i, x_{i+1})$. Define

$\hat{D}\colon X^2 \to [0, \infty), \hat{D}(x, y) = \inf\{\mathfrak{l}(s)\colon s \text{ is an } \varepsilon\text{-chain between } x \text{ and } y\}.$

Theorem (<- - - - - - - -)

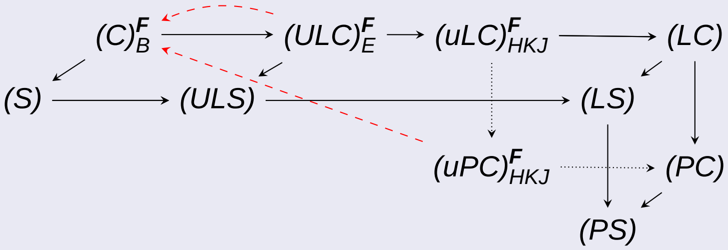
Assume $\langle X, d\rangle$ is connected.

- *For any $\varepsilon > 0$ there is an ε-chain between any two points.*
- *$\hat{D}$ is a metric topologically equivalent to d.*
- *If $\langle X, d\rangle$ is complete, than so is $\langle X, \hat{D}\rangle$.*
- *If $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (ULC), then $f\colon \langle X, \hat{D}\rangle \to \langle X, \hat{D}\rangle$ is (C).*
- *If $\langle X, d\rangle$ is also compact and $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (ULS), then $f\colon \langle X, \hat{D}\rangle \to \langle X, \hat{D}\rangle$ is (S).*

Fixed and Periodic Points - Connected Spaces

Theorem (Rectifiably Path Connected Spaces)

Assume X is complete and rectifiably path connected. No combination of any of the properties shown imply any other property, unless the graph forces such implication. Neither does any combination imply the exitance of a periodic point unless it contains (C), (ULC), (uLC) or (uPC).



Fixed and Periodic Points - Connected Spaces

Definition

A metric space $\langle X, d\rangle$ is *d-convex* provided for any distinct points $x, y \in X$ there exists a path $p\colon [0,1] \to X$ from x to y such that

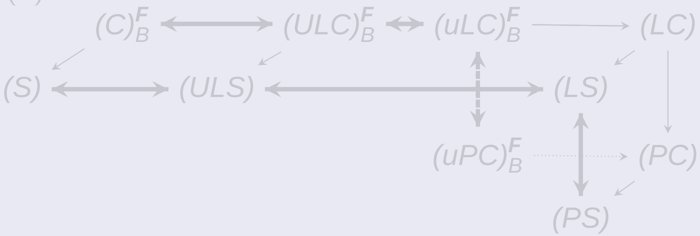
$$d(p(t_1), p(t_3)) = d(p(t_1), p(t_2)) + d(p(t_2), p(t_3))$$

whenever $0 \leq t_1 < t_2 < t_3 \leq 1$.

Fixed and Periodic Points - Connected Spaces

Theorem (d-convex Spaces)

Assume X is complete and d-convex. Jungck (1982) showed (uPC) $\Rightarrow$ (C) *with the same* λ*. A modified argument shows that* (PS) $\Rightarrow$ (S).



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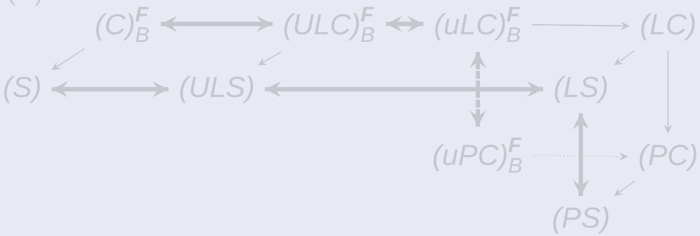

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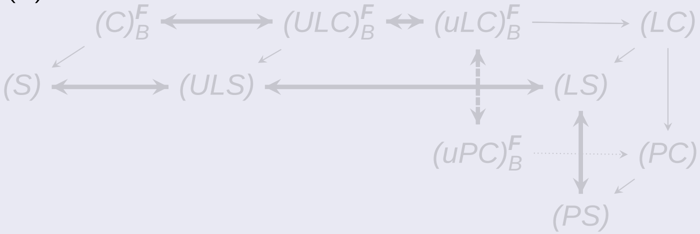

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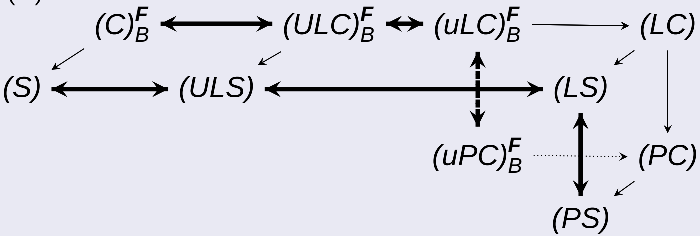

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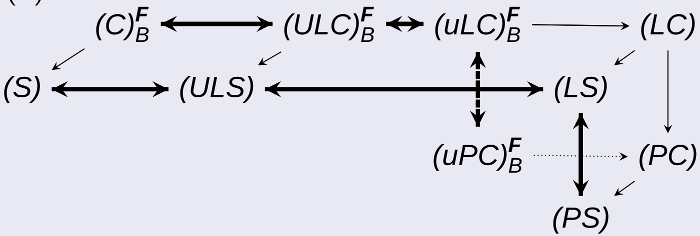

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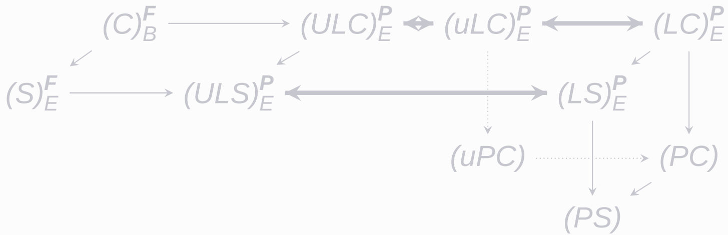
Fixed and Periodic Points - Compact Spaces

Theorem (Compact Spaces)

Assume $\langle X, d\rangle$ is compact. Ding and Nadler (2002) and C&J 2015 showed $(\mathrm{LC}) \Rightarrow (\mathrm{ULC})$ *and* $(\mathrm{LS}) \Rightarrow (\mathrm{ULS})$*.*



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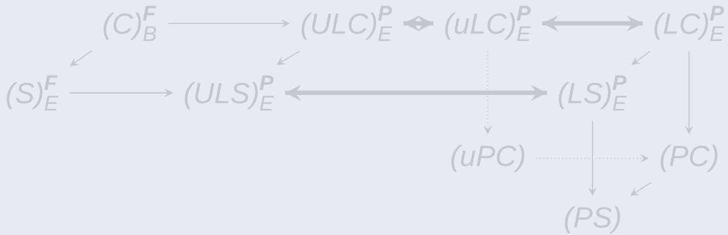

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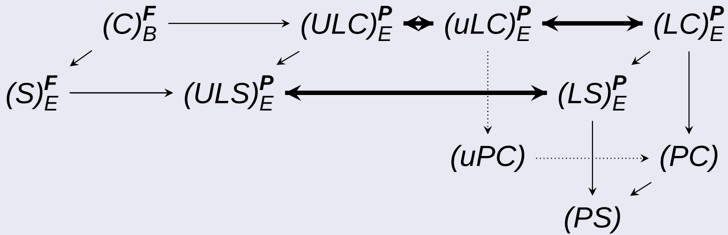

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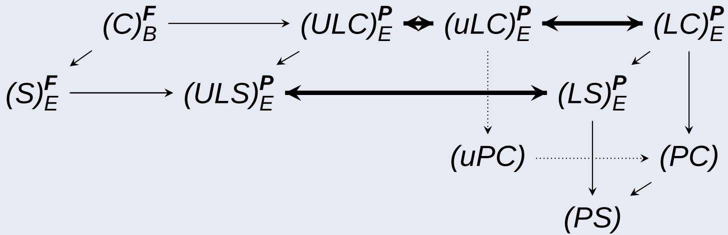

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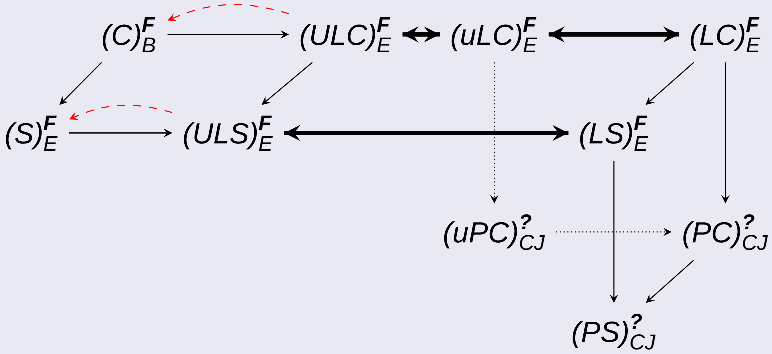
Assume X is compact and connected.



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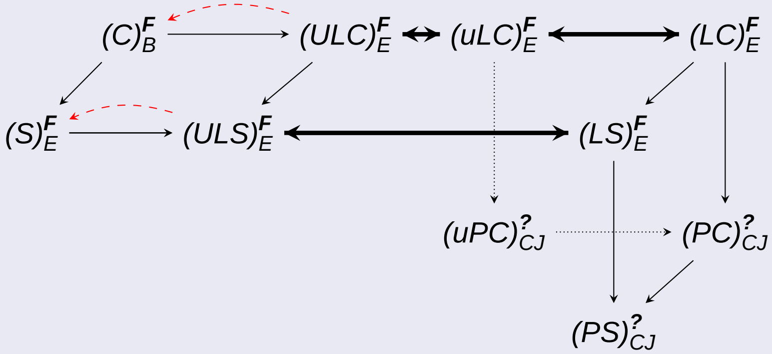

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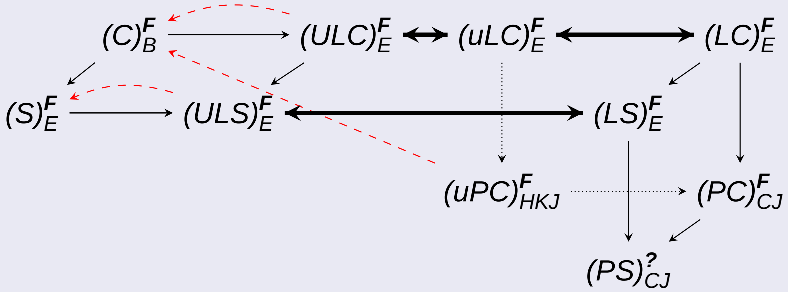

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Fixed and Periodic Points - Compact Spaces

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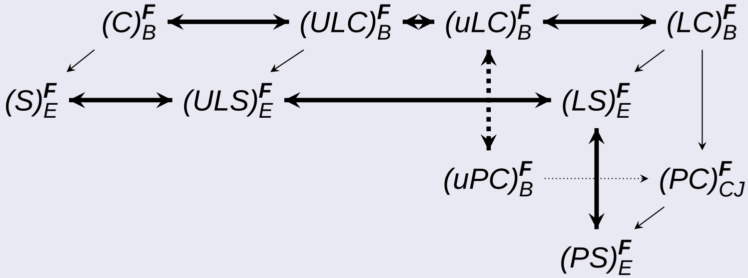

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Open Problems

1. Assume that $\langle X, d\rangle$ is compact and either connected or path connected. If the map $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (PS), must f have either fix or periodic point? What if f is (PC)? or (uPC)?
2. Assume that $\langle X, d\rangle$ is compact and rectifiably path connected. If the map $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (PS), does it imply that f has a fixed or periodic point?

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Thank you
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Open Problems

3. Let $X \subset \mathbb{R}$ be compact perfect and let g be a function from X onto X^2. Can g be differentiable?

If a differentiable $g = \langle f, h\rangle$ as in Problem 3 existed then $f : X \to X$ would be a surjection with $f'(x) = 0$ except for a meager subset of X, [C&J, 2014].

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There exists a perfect compact set $\mathfrak{X} \subseteq \mathbb{R}$ and autohomeomorphism $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{X}$ with $\mathfrak{f}'(x) = 0$ for all $x \in X$. It follows that $\mathfrak{f}$ is $\lambda - (\mathrm{uPC})$ with any $\lambda \in [0, 1)$. Moreover, $\langle \mathfrak{X}, \mathfrak{f} \rangle$ is a minimal dynamical system so $\mathfrak{f}$ has no periodic points.

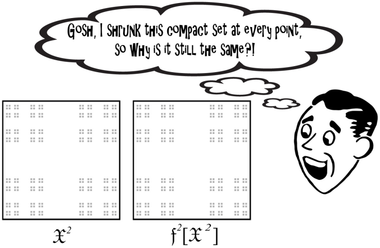

Figure: Action of $\mathfrak{f}^2 = \langle \mathfrak{f}, \mathfrak{f} \rangle$ on $\mathfrak{X}^2$.

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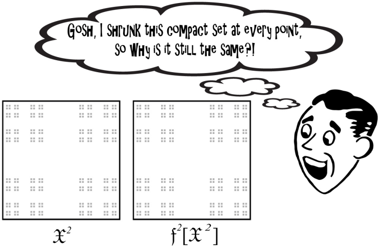

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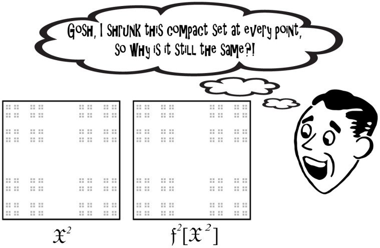

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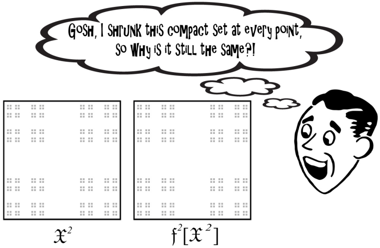

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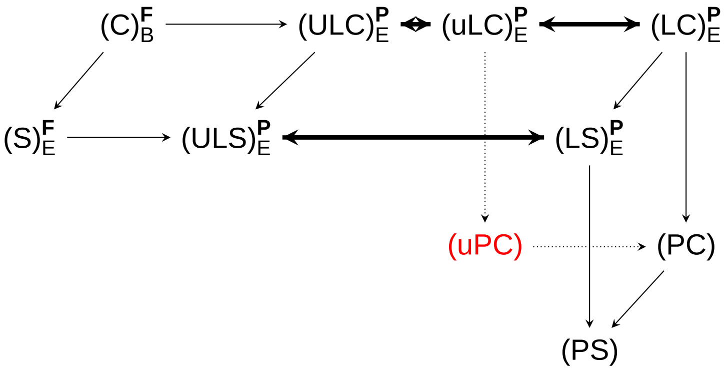
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(PC)
(PS)