Generic norms and metrics on countable Abelian groups

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Plan of the talk

Objective

For a fixed countable Abelian group G, the goal is to investigate the Polish space of all invariant metrics on G and look for properties of metrics that hold generically. That is, we look for properties such that all but meager-many metrics satisfy those properties.

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- We shall look for a metric space such that the generic metric is isometric to it.
- We shall look for an Abelian Polish metric group such that the completion of the generic metric is isometrically isomorphic to it.
- We shall apply these results to the universal Abelian Polish groups.

Introduction

Let G be an Abelian group. A metric d on G is invariant if

$$\forall a, b, c \in G \; (d(a, b) = d(a + c, b + c)).$$

Or equivalently,

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A function $\lambda : G \to \mathbb{R}_0^+$ is a norm (value) if it satisfies

- $\lambda(g) = 0$ iff $g = 0$, for every $g \in G$;
- $\lambda(g) = \lambda(-g)$, for every $g \in G$;
- $\lambda(g + h) \leq \lambda(g) + \lambda(h)$, for every $g, h \in G$.

There is a one-to-one correspondence between invariant metrics and norms on Abelian groups.

Fact

A topological Abelian group is metrizable iff it is metrizable by an invariant metric.
In general, topology on an Abelian topological group is determined by a family of invariant pseudometrics.

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Theorem (Melleray, Tsankov)

Let G be a countable unbounded Abelian group. Then the set $\{d \in \mathcal{M}_G : (G, d) \text{ is extremely amenable}\}$ is dense G_δ in $\mathcal{M}_G$.

Some definitions

The *Urysohn universal metric space* $\mathbb{U}$ is the unique separable complete metric space with the following properties:

- it contains an isometric copy of every separable metric space,
- any partial isometry between two finite subsets extends to an autoisometry of the whole space.

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The *rational* Urysohn universal metric space $\mathbb{QU}$ is the unique countable metric space with rational distances having the following properties:

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Fact

The completion of $\mathbb{QU}$ is $\mathbb{U}$.

Theorem (Cameron, Vershik)

There exists an invariant metric d on $\mathbb{Z}$ such that $(\mathbb{Z}, d)$ is isometric to the rational Urysohn space.
In particular, the Urysohn space has a structure of a monothetic group.

Group structures on the Urysohn space

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Theorem(Shkarin; Niemiec)

There exists an Abelian Polish metric group that is topologically universal Abelian Polish group.
It has a distinguished countable dense subgroup which is algebraically isomorphic to $\bigoplus_{\mathbb{N}} \mathbb{Q}/\mathbb{Z}$ and isometric to $\mathbb{QU}$.

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- the set $\{d \in \mathcal{M}_G : \overline{(G, d)} \text{ is isometric to } \mathbb{U}\}$ is G_δ;
- the set $\{d \in \mathcal{M}_G : (G, d) \text{ is isometric to } \mathbb{QU}\}$ is dense.

Thus the set $\{d \in \mathcal{M}_G : \overline{(G, d)} \text{ is isometric to } \mathbb{U}\}$ is comeager in $\mathcal{M}_G$.

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Theorem & Conjecture (Niemiec)

No Abelian group of exponent 3 is isometric to $\mathbb{QU}$ or $\mathbb{U}$.
A conjecture is that the same is true for other exponents greater than 3.

Generic structures

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Fact

Comeager many countable metric spaces have completions isometric to the Urysohn space.

Extreme amenability

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Recall from introduction.

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Main result

Theorem

Let G be a countable Abelian group satisfying $G \cong \bigoplus_{\mathbb{N}} G$. Then there is a comeager set $\mathcal{G} \subseteq \mathcal{M}_G$ such that for every $d, p \in \mathcal{G}$, the complete groups $\overline{(G, d)}$ and $\overline{(G, p)}$ are isometrically isomorphic.

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Corollary

Let G be a countable unbounded Abelian group such that $G \cong \bigoplus_{\mathbb{N}} G$. Then there is a comeager set $\mathcal{G} \subseteq \mathcal{M}_G$ such that for every $d, p \in \mathcal{G}$, the complete groups $\overline{(G, d)}$ and $\overline{(G, p)}$ are isometrically isomorphic groups that are extremely amenable and isometric to the Urysohn universal space.

Universal Polish Abelian groups

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The universal Abelian Polish groups are unique and extremely amenable (and isometric to the Urysohn space).

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The metrics $\rho \in \mathcal{M}_{\sum_{\mathbb{N}} \mathbb{Q}/\mathbb{Z}}$ that satisfy that $\overline{(\sum_{\mathbb{N}} \mathbb{Q}/\mathbb{Z}, \rho)}$ is universal and homogeneous are those that satisfy the ε-one-point extension property. That is a G_δ condition. The Shkarin's construction shows that those metrics form a dense subset of $\mathcal{M}_{\sum_{\mathbb{N}} \mathbb{Q}/\mathbb{Z}}$.

Finally, we intersect these comeager sets of metrics to get

- that all those universal Polish groups are isometrically isomorphic (the main theorem);
- they are extremely amenable (Melleray-Tsankov);
- they are isometric to the Urysohn space.

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Investigate the Polish space of all bi-invariant metrics on countable (non-abelian) groups.

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Theorem

In the Polish space of all bi-invariant metrics on F_∞, the free group of countably many free generators, bounded by 1 there is a comeager subset of metrics whose completion is isometrically isomorphic to the universal Polish metric group with bi-invariant metric bounded by 1.

Problems

Does there exist a natural Polish space of

- all left-invariant metrics on a countable group?
- all left-invariant uniformly discrete metrics?
- all left-invariant uniformly discrete proper metrics?

A straightforward computation gives that they are $F_{\sigma\delta}$ subsets of $\mathbb{R}^{G\times G}$.

Some references

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