The Zariski topology of a group

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Markov introduced in 1944 the notion of an *algebraic set* of a group G as an arbitrary intersection of finite unions of solution-sets of appropriately defined one variable equations over G. He called *unconditionally closed* a subset of G that is closed in any Hausdorff group topology on G, observing that algebraic sets are unconditionally closed. He proved that these two notions coincide for countable groups and raised the question on whether this remains true in general. One can define two T_1 topologies $\mathfrak{Z}_G$ (*Zariski topology*) and $\mathfrak{M}_G$ (*Markov topology*) on G having as all closed sets all algebraic, and all unconditionally closed sets, respectively. Then $\mathfrak{Z}_G \leq \mathfrak{M}_G$ and Markov's question is equivalent to asking whether $\mathfrak{Z}_G = \mathfrak{M}_G$. The equality $\mathfrak{Z}_G = \mathfrak{M}_G$ for Abelian G was proved by Perel'man, while a counterexample in the general case is attributed to Hesse, but both results were never published. Groups G with discrete $\mathfrak{M}_G$, answering Markov problem on the existence of infinite non-topologizable group, were built by Shelah (under CH), while infinite groups G with discrete $\mathfrak{Z}_G$ were built (in ZFC) by Ol'shankij and his school.

The aim of the talk is to present recent applications of these topologies:

1. some joint results with D. Shakhmatov concerning other problems of Markov for Abelian groups (among them, the description of the potentially dense subsets, a positive solution of Markov's conjecture on connected topologization, etc.);

2. some recent results of Banakh, Chang, Gartside, Glyn, Guran, Megrelishvili, Polev, Protasov, Sipacheva and Toller in the non-Abelian case.

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